Lecture notes for Math 221, Winter 2013

Lenya Ryzhik*

February 28, 2013

Essentially nothing found here is original except for a few mistakes and misprints here and there. These lecture notes are based on the book "Image Processing and Analysis" by Tony Chan and Jackie Shen.

1 Image representations

In order to start talking about image processing we first need to describe ways to store, encode and interpret images. Imaging is part science, part art: there is no best way to represent all images, as each image has its own key features that need to be represented well, but there are large classes of images that can be efficiently represented in similar fashion, and that is going to be our focus here.

We will mostly deal with idealized unpixelized images that we will consider as a function (physical image) $u(x, y)$ of two variables defined in an image window that will typically be an (open) rectangle

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x \in (a, b), y \in (c, d)\}.$$

We usually deal not with an image directly but with its representation (or transform $\mathcal{T}$) that allows us to store it. Mathematically, this means that we have access to (or choose to use) the transform $w = \mathcal{T}u$ that maps the class of physical images U to its range W that is called the transform space.

A representation is lossless if any physical image $u \in U$ can be reconstructed exactly from its representation $w = \mathcal{T}u$. That is, there exists a reconstruction transform $\mathcal{R}$ that maps W to U in a one-to-one way. In other words, two different images u_1 and u_2 have two different representations $w_1 = \mathcal{T}u_1$ and $w_2 = \mathcal{T}u_2$. In that case we have a well-defined reconstruction formula $u = \mathcal{R}(\mathcal{T}u)$. In the imaging jargon the representation $\mathcal{T}$ is called the analysis transform, and the reconstruction map $\mathcal{R}$ is called the synthesis transform, since it is used to synthesize the image from the recording.

Lossless transforms offer best quality as you can recover the original image exactly but are both expensive in terms of memory – after all, you do not lose any information, – and often useless and unappreciated – in the age when most images are viewed either on a (relatively small) computer screens, or, G-d forbid, a tiny iPhone screen, only the main visually important visual features need to be reconstructed, and this is what the best lossy algorithms do efficiently.

*Department of Mathematics, Stanford University, Stanford CA 94305, USA; ryzhik@math.stanford.edu

If the representation is lossy, meaning that there exist two images u_1 and u_2 with the same representation $w = \mathcal{T}u_1 = \mathcal{T}u_2$, we obviously can not recover the image exactly from its representation via the synthesis transform: the task is to design the synthesis transform in a way that would recover "the best" of the candidates u_1 and u_2 (or possibly an infinite number of candidate physical images that all have the same representation). This is where decisions are made – and where science meets art.

We now describe several image spaces we will use – the difference here is not purely mathematical. As we will see, same images (functions) may be very close to each other or very far from each other, depending on the way we measure the distance between images, or, in other words, depending on the space of functions we consider. For example, consider the functions

$$u(x) = \begin{cases} 0, & x \text{ outside } [0,1] \\ 1, & x \in [0,1], \end{cases}$$

and

$$v_n(x) = \begin{cases} 0, & x \text{ outside } [1/n,1] \\ 1, & x \in [1/n,1]. \end{cases}$$

If we measure the distance in the sup-norm, the distance between these functions is one for all n:

$$\sup_{x\in\mathbb{R}} |u(x) - v_n(x)| = 1.$$

On the other hand, these functions look very similar to each other for all n – so if the very precise location of the jump is not terribly important then the function $v_n(x)$ (for n large) is a very good approximation for the function $u(x)$. Therefore, depending on our needs, we will measure the distance in different ways.

1.1 Images as distributions

The class of generalized functions or distributions is probably the largest one can consider. It is also necessary if we would like to consider images that can not be described by ordinary functions, such as (i) a hot spot – a very bright spot idealized to be localized at a single point, which we can take to be at the origin $(x, y) = (0, 0)$, (ii) a uniform bright line which we can take, for instance, to be $\{x_1 = 0\}$, or (iii) a non-uniformly bright line that is localized at $x_1 = 0$ but whose intensity depends on the vertical coordinate x_2.

In order to define what a generalized function is, we consider the set of test functions

$$D(\Omega) = \{\phi \in C^\infty(\Omega), \quad \text{supp } \phi \subset \Omega\},$$

that is, the support of u is a compact subset of Ω that is contained strictly inside Ω. A generalized function (or a distribution) u is a linear functional on $D(\Omega)$, that is, for any test function $\phi \in D(\Omega)$ the value $\langle u, \phi\rangle$ is defined, and this functional is linear:

$$\langle u, a\phi + b\psi\rangle = a\langle u, \phi\rangle + b\langle u, \psi\rangle, \quad \text{for all } a, b \in \mathbb{R}.$$

Going back to our examples, we can define a bright spot at a point $p = (a, b) \in \Omega$ as the delta function $u = \delta(x - p)$, $x = (x_1, x_2)$. This is a distribution such that $\langle u, \phi\rangle = \phi(a, b)$ for

any test function $\phi \in D(\Omega)$. A uniform bright line $\{x_1 = a\}$ can be defined as a distribution v such that

$$\langle v, \phi \rangle = \int_{\mathbb{R}} \phi(a, x_2) dx_2,$$

for any test function $\phi \in D(\Omega)$. Finally, a non-uniform bright line $\{x_1 = a\}$, with an intensity distribution $r(x_2)$ can be described as the distribution w such that

$$\langle w, \phi \rangle = \int_{\mathbb{R}} r(x_2)\phi(a, x_2) dx_2.$$

Exercise 1.1 Show that these definitions make sense. Consider a sequence of classical (albeit discontinuous) functions

$$u_n(x) = \begin{cases} n^2/\sqrt{\pi}, & |x| \le 1/n, \\ 0, & \text{otherwise}, \end{cases}$$
$$v_n(x) = \begin{cases} n/2, & |x_1| \le 1/n, \\ 0, & \text{otherwise}, \end{cases}$$
$$w_n(x) = \begin{cases} b(x_2)n/2, & |x_1| \le 1/n, \\ 0, & \text{otherwise}, \end{cases}$$

that intuitively approximate a bright spot at $p = (0, 0)$, a uniform bright line, and a bright line of a non-uniform intensity, both along $\{x_1 = 0\}$, and show that for any test function $\phi \in D(\Omega)$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} u_n(x)\phi(x)dx = \phi(0, 0),$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^2} v_n(x)\phi(x)dx = \int_{\mathbb{R}} \phi(0, x_2)dx_2,$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^2} w_n(x)\phi(x)dx = \int_{\mathbb{R}} b(x_2)\phi(0, x_2)dx_2.$$

In practice, images can usually be thought of as positive functions, and this extends to their representation as generalized functions.

Definition 1.2 *A distribution u is positive if for any non-negative test function $\phi \in D(\Omega)$ we have $\langle u, \phi \rangle \ge 0$.*

This greatly reduces the class of distributions – for instance, the value of $\langle u, \phi \rangle$ can not depend on the derivatives of ϕ, as can be seen from the following simple observation.

Theorem 1.3 *Let u be a non-negative distribution on Ω. Then for any compact subset K of Ω there exists a constant C_K so that for any test function ϕ supported in K we have*

$$|\langle u, \phi \rangle| \le C_K \|\phi\|_\infty. \tag{1.1}$$

The proof is not difficult: a basic fact from the real analysis is that given any compact set K there exists a smooth non-negative function ψ that is supported in Ω and $\psi(x) = 1$ for all $x \in K$. Then, for any test function $\phi \in D(\Omega)$ supported in K, the function

$$\zeta(x) = \|\phi\|_\infty \psi(x) - \phi(x)$$

is non-negative. As the distribution u is non-negative, it follows that $\langle u, \zeta \rangle \geq 0$, which is equivalent to

$$\langle u, \phi \rangle \leq \|\phi\|_\infty \langle u, \psi \rangle.$$

Replacing ϕ by $(-\phi)$ we get the inequality (1.1) with $C_K = \langle u, \psi \rangle$.

Let us now discuss how to quantify the information content of an image understood as a distribution. Human vision is essentially based on observing variations in patterns and not the absolute value of patterns[1]. Therefore, in order to understand the information that is contained in the image $u(x)$ in an open set $U \subset \Omega$, we can restrict ourselves to test functions ϕ supported inside U and of mean zero:

$$\int_U \phi(x)dx = 0, \tag{1.2}$$

and consider

$$I_u = \sup \Big\{ \langle u, \phi \rangle : \ \phi \in D(\Omega), \ \ \text{supp } \phi \subset U, \ \ \|\phi\|_\infty \leq 1, \ \ \int_U \phi(x)dx = 0 \Big\}.$$

For example, $I_u = 0$ for any image $u(x) \equiv \text{const}$ – as we wish for reasons explained above. We can substitute the mean zero test functions by those which are the divergence of a vector field $g = (g_1, g_2)$ with $g_{1,2} \in D(\Omega)$ and supported in U:

$$I'_u = \sup \Big\{ \langle u, \phi \rangle : \ \phi \in D(\Omega), \ \ \phi = \nabla \cdot g, \ \ g = (g_1, g_2), \ \ \text{supp } g_{1,2} \subset U, \ \ \|g_{1,2}\|_\infty \leq 1 \Big\}. \tag{1.3}$$

Note that if $\phi = \nabla \cdot g$ with $g = 0$ on ∂U then ϕ satisfies (1.2). Using the Stokes' formula

$$\int_\Omega u(\nabla \cdot g)dx = -\int_U (\nabla u \cdot g)dx,$$

we see that (1.3) is nothing but the total variation of u:

$$TV_U(u) = \sup \Big\{ \langle u, \phi \rangle : \ \phi \in D(\Omega), \ \ \phi = \nabla \cdot g, \ \ g = (g_1, g_2), \ \ \text{supp } g_{1,2} \subset U, \ \ \|g_{1,2}\|_\infty \leq 1 \Big\}. \tag{1.4}$$

This is the first time in these notes that the total variation norm of an image appears – it will show up very frequently from now on, as it plays a very important role in this subject.

If we know that an image u is not just a distribution but an $L^p(U)$ function, we can define its average over a set U as

$$\langle u \rangle_U = \frac{1}{|U|} \int_U u(x)dx,$$

and define the information content of u on U as its p-mean oscillation

$$\sigma_p(u|U) = \left(\frac{1}{|U|} \int_U |u - \langle u \rangle_U|^p dx \right)^{1/p}.$$

[1]For instance, a skier in flat light does not care if the uniform color he sees has any particular shade of white or grey – he wants to see the variations in the terrain!

1.2 Images as functions of bounded variation

Working with the space of distributions is not very convenient from the practical point of view as this space is huge and includes many functions we would never consider as an image. In addition, it is not a Banach space, so solving optimization problems in this space maybe unnecessarily complicated. Therefore, it is important to have a sub-class of distributions that includes "mostly nice" functions but is not too restrictive so that it contains sufficiently rough functions that correspond to real world images. A convenient space for that purpose is the space of BV functions. An L^1 function u belongs to the class $BV(\Omega)$ if its gradient (in the sense of distributions) satisfies

$$TV(u) := \sup\{\langle \nabla u, g\rangle : \ g = (g_1, g_2), \quad g_{1,2} \in D(\Omega), \quad \|g\|_\infty \le 1\} < +\infty. \tag{1.5}$$

This is a Banach space with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + TV(u).$$

An example of a BV function is an L^1-function with a gradient that is also in L^1 – this is the Sobolev class $W^{1,1}(\Omega)$. This is because we have the inequality

$$|\langle \nabla u, g\rangle| \le \|\nabla u\|_{L^1}\|g\|_\infty,$$

so that the supremum in (1.5) is finite. An important difference between $W^{1,1}$ functions and BV is that the former may not have sharp edges – discontinuity along a curve, and the latter can. For example, consider a function $u(x) = H(x_1)$, where $H(x)$ is the Heaviside function, defined on the square $\Omega = (-1, 1) \times (-1, 1)$. Its distributional gradient is $\nabla u(x) = (\delta(x_1), 0)$ which is not an L^1-function but it is in $BV(\Omega)$ since

$$|\langle \nabla u, g\rangle| = \left| \int_{-1}^{1} g_1(0, x_2)dx_2 \right| \le 2\|g\|_\infty.$$

Allowing edges is critical for good image representations as they are present in most real world images. Of course, L^p functions also can have edges, the advantage of working with BV functions is that the $\|u\|_{BV}$ takes into account the number of edges while the L^p-norm does not: each edge "costs" in the BV-norm but not in the L^p-norm.

Let us now recall some basic properties of BV functions as we will extensively use them.

Theorem 1.4 *(L^1-lower semicontinuity) Assume that $u_n \to u$ in $L^1(\Omega)$. If u_n are uniformly bounded in BV (there exists M so that $\|u_n\|_{BV} \le M$ for all n), then u is BV as well and $\|u\|_{BV} \le M$. Moreover, we have*

$$\int_\Omega |\nabla u| \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|. \tag{1.6}$$

The proof is not long. Let

$$g = (g_1, g_2) \in D(\Omega) \text{ with } \|g\|_\infty \le 1, \tag{1.7}$$

then

$$\langle \nabla u, g\rangle = -\int_\Omega u(\nabla \cdot g)dx = -\lim_{n\to\infty} \int_\Omega u_n(\nabla \cdot g)dx, \tag{1.8}$$

as $u_n \to u$ in $L^1(\Omega)$, so that

$$\left| \int_\Omega u(\nabla \cdot g)dx - \int_\Omega u_n(\nabla \cdot g)dx \right| \leq \|u - u_n\|_{L^1(\Omega)} \|\nabla \cdot g\|_{L^\infty(\Omega)}.$$

However, as each u_n is BV and from properties (1.7) of $g(x)$ we conclude that

$$-\int_\Omega u_n(\nabla \cdot g)dx \leq \int_\Omega |\nabla u_n|.$$

Hence, (1.8) implies that

$$TV(u) \leq \liminf_{n\to\infty} TV(u_n),$$

and, since $\|u_n\|_{L^1} \to \|u\|_{L^1}$, we deduce that

$$\|u\|_{BV} \leq \liminf_{n\to\infty} \|u_n\|_{BV},$$

which finishes the proof.

It follows from Theorem 1.4 that BV(Ω) is a Banach space. Indeed, let u_n be a Cauchy sequence in BV(Ω). The definition of the BV norm implies that u_n is also Cauchy in $L^1(\Omega)$. Let u be its limit in $L^1(\Omega)$ (which we know is a Banach space), then for each m fixed we have, from Theorem 1.4

$$\int_\Omega |\nabla u_m - \nabla u| \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_m - \nabla u_n|.$$

The right side goes to zero as $m \to +\infty$ since u_n is a Cauchy sequence in BV(Ω). Therefore, the sequence u_m also converges to u in BV(Ω). Hence, BV(Ω) is a Banach space.

One can show that BV(Ω) functions have a trace on the boundary $\partial\Omega$, as do functions from $W^{1,1}(\Omega)$. It is reasonable to ask how close the spaces $W^{1,1}(\Omega)$ and BV(Ω) are. First, one can prove the following approximation result.

Theorem 1.5 *For any function $u \in BV(\Omega)$, one can find a sequence of smooth approximations $u_n \in C^\infty(\Omega)$, which all have the same trace $g_n \equiv g$, such that $u_n \to u$ in $L^1(\Omega)$ and*

$$\int_\Omega |\nabla u_n| dx \to \int_\Omega |\nabla u|. \tag{1.9}$$

We can not hope for a better result than this, that is, we can not expect that for any $u \in$ BV(Ω) we can find a sequence $u_n \in C^\infty(\Omega)$ so that

$$\int_\Omega |\nabla u_n - \nabla u| \to 0,$$

as $n \to \infty$. Indeed, let $\Omega = [-1, 1] \times [-1, 1]$ and $u(x_1, x_2) = H(x_1)$, the Heaviside function, so that $\nabla u = \delta(x_1)$. Then, if such approximation were to exist, we would have a sequence of smooth functions $w_n = \partial u_n/\partial x_1$ that would approximate $\delta(x_1)$ in the L^1-sense, which is impossible.

Still, Theorem 1.6 provides a convenient tool to transplant many of the results on $W^{1,1}(\Omega)$-functions to BV(Ω)-functions. Let $u \in$ BV(Ω) and $f = T(u)$ be its trace. We say that a

functional $L(u,f)$ on $\mathrm{BV}(\Omega)$ is L^1-lower semicontinuous if for any sequence $u_n \in \mathrm{BV}(\Omega)$ with $u_n \to u$ in $L^1(\Omega)$ and such that the traces of all u_n are the same: $f_n = T(u_n) \equiv f = T(u)$, one has

$$L(u,f) \le \liminf_{n\to\infty} L(u_n,f).$$

Corollary 1.6 *Let $L(u,f)$ be an L^1-lower semi-continuous functional in $BV(\Omega)$ and that $E(t)$ is a continuous function of t. If*

$$L(u,f) \le E(|\nabla u|(\Omega)) \tag{1.10}$$

for any $u \in W^{1,1}(\Omega)$ then the same inequality holds for all $u \in BV(\Omega)$.

Indeed, let u_n be the approximating sequence from Theorem 1.5. The lower semicontinuity property of $L(u,f)$ and L^1-convergence of u_n to u imply that

$$L(u,f) \le \liminf_{n\to\infty} L(u_n,f). \tag{1.11}$$

On the other hand, the bound (1.10) implies that

$$L(u_n,f) \le E(|\nabla u_n|(\Omega)). \tag{1.12}$$

Finally, it follows from Theorem 1.5 and continuity of the function $E(t)$ that

$$E(|\nabla u_n|(\Omega)) \to E(|\nabla u|(\Omega)). \tag{1.13}$$

Corollary 1.6 follows from the last three inequalities.

Another consequence of Theorem 1.5 is crucial to establish existence of solutions to various variational problems used in imaging.

Theorem 1.7 *Let Ω be a Lipschitz domain, and u_n be a bounded sequence in $BV(\Omega)$. Then u_n has a subsequence that converges in $L^1(\Omega)$.*

For the proof, we use Theorem 1.5 to find functions $w_n \in C^\infty(\Omega)$ such that

$$\int_\Omega |w_n - u_n| dx \le \frac{1}{n},$$

and

$$\int_\Omega |\nabla w_n| dx \le \int_\Omega |\nabla u_n| + 1.$$

As u_n are uniformly bounded in $\mathrm{BV}(\Omega)$, it follows that w_n is uniformly bounded in $W^{1,1}(\Omega)$. Sobolev embedding theorem (of $W^{1,1}(\Omega)$ into $L^1(\Omega)$) implies that w_n has a subsequence w_{n_k} that converges in $L^1(\Omega)$ to a function w. The triangle inequality implies that

$$\|u_{n_k} - w\|_{L^1(\Omega)} \le \|u_{n_k} - w_{n_k}\|_{L^1(\Omega)} + \|w_{n_k} - w\|_{L^1(\Omega)} \le \frac{1}{n} + \|w_{n_k} - w\|_{L^1(\Omega)},$$

whence u_{n_k} also converges to w in $L^1(\Omega)$.

The co-area formula

The co-area formula rephrases the total variation of a function in terms of its level sets, connecting two notions that are crucial in many imaging applications.

Let us start with a smooth function $u(x, y)$ and assume that $\nabla u(p) \neq 0$ for some $p = (x, y) \in \Omega$. Without loss of generality we may further assume that $u_y(p) \neq 0$. The implicit function theorem implies that locally near p we can 'solve for y": for each λ close to $\lambda_0 = u(p)$ there exists a level curve $y = y(x, \lambda)$ so that $u(x, y(x, \lambda)) = \lambda$. We may further parametrize the level set $(x, y(x, \lambda))$ by the arclength s:

$$x = x(s), \quad y = y(x(s), \lambda).$$

This gives the local coordinates

$$x = x(s, \lambda), \quad y = y(s, \lambda),$$

keeping the identity

$$u(x(s, \lambda), y(s, \lambda)) = \lambda.$$

Differentiating with respect to s and λ gives

$$0 = (x_s, y_s) \cdot \nabla u, \tag{1.14}$$

and

$$1 = (x_\lambda, y_\lambda) \cdot \nabla u. \tag{1.15}$$

Moreover, as s is the arclength parameter, we have

$$x_s^2 + y_s^2 = 1.$$

The last two identities imply that the vectors ∇u and (x_s, y_s) do not vanish. It follows then from (1.14) that ∇u is parallel to the unit vector $(-y_s, x_s)$:

$$\nabla u = \pm |\nabla u|(-y_s, x_s).$$

Inserting this into (1.15) gives

$$1 = \pm |\nabla u|(-x_\lambda y_s + y_\lambda x_s). \tag{1.16}$$

The expression in the parenthesis is simply the Jacobian of the map $(s, \lambda) \to (x, y)$, that is, we have:

$$|\nabla u| dx dy = ds d\lambda, \tag{1.17}$$

or, in other words (assuming that the change of variables is global),

$$\int_\Omega |\nabla u| dx dy = \int_{-\infty}^{\infty} \left(\int_0^{|\gamma_\lambda|} ds \right) d\lambda, \tag{1.18}$$

with the integral in the right side taken over all possible values of λ, and s varying from $s = 0$ to $s = |\gamma_\lambda|$, the arclength of the level set curve $\gamma_\lambda = \{u(x, y) = \lambda\}$. We can integrate the s-variable out in (1.18) to obtain

$$\int_\Omega |\nabla u| dx dy = \int_{-\infty}^{\infty} |\gamma_\lambda| d\lambda, \tag{1.19}$$

which is the co-area formula for smooth functions.

The above argument does not quite apply for general BV functions for the simple reason that we need to understand first what we understand by a level set γ_λ of a (possibly discontinuous) BV function. The second issue is that the level set of a BV function may be terribly irregular – what do we mean by its arclength?

The way around these issues is to define the super-level set (or a cumulative level set)

$$E_\lambda = \{(x,y) \in \Omega : \quad u(x,y) < \lambda\}.$$

If the function $u(x,y)$ is smooth then the level set $\gamma_\lambda = \partial E_\lambda$. In general we define the perimeter of E_λ as

$$\mathrm{Per}(E_\lambda) = \int_\Omega |\nabla \chi_{E_\lambda}|,$$

that is, the total variation of the indicator function of E_λ. Then we have the following result.

Theorem 1.8 *(The co-area formula) Let $u \in BV(\Omega)$, then*

$$\int_\Omega |\nabla u| = \int_{-\infty}^{\infty} \mathrm{Per}(E_\lambda) d\lambda. \tag{1.20}$$

We are not going to present the rigorous proof here – a careful treatment can be found in the book by Evans and Gariepy.

Notice that (1.20) can be nicely interpreted for edges: an edge is essentially a very thin domain Ω where the function takes "many" values, and the perimeter of each level set is $O(1)$ – the length of the edge, making the right side of (1.20) "anomalously large" relative to the size of the small neighborhood around the edge. Hence, the TV measure of u in a narrow domain near the edge is much larger than the domain volume – the right side of the co-area formula (1.20) is $O(1)$ even if Ω has a small volume in such case. This means that the TV measure can detect edges in this way automatically.

1.3 Level-set representation

As we have started discussing level sets, it is appropriate to recall the level set representation for functions developed by Osher, Sethian, Fedkiw and others in the context of imaging.

Let u be a sufficiently smooth function. For each real number λ, a level set of $u(x)$ is

$$\gamma_\lambda = \{x \in \Omega : \ u(x) = \lambda\}.$$

The level sets are disjoint: $\gamma_\lambda \cap \gamma_\mu = \emptyset$ for $\lambda \neq \mu$, and the whole domain Ω is partitioned into the level sets:

$$\Omega = \bigcup_{\lambda \in \mathbb{R}} \gamma_\lambda.$$

A value λ is regular if the gradient ∇u does not vanish anywhere on γ_λ. In that case, the level set γ_λ is a regular one-dimensional curve (that may consist of several components). Otherwise, λ is singular, and a point x_0 where $\nabla u(x_0) = 0$ is called a critical point.

When we deal with non-smooth functions, we consider instead the cumulative level sets

$$F_\lambda(u) = \{x \in \Omega : \ u(x) \leq \lambda\}.$$

Given a function u we have a one-parameter family of cumulative level sets $\{F_\lambda(u),\ \ \lambda \in \mathbb{R}\}$. This is what we called previously the analysis step – we record the function as a family of cumulative level sets. How can we do the synthesis, that is, recover the function u from its cumulative level sets? The synthesis is simple: assume we are given a family of sets F_λ that satisfy the following compatibility conditions: (i) for any $\lambda \leq \mu$ we have $F_\lambda \subseteq F_\mu$, and

$$F_\lambda = \bigcap_{\mu > \lambda} F_\mu,$$

and (ii) the set

$$F_{-\infty} = \bigcap_{\lambda \in \mathbb{R}} F_\lambda$$

is empty while

$$\Omega = \bigcup_{\lambda \in \mathbb{R}} F_\lambda.$$

Then we set

$$u(x) = \inf\{\mu :\ x \in F_\mu\}. \tag{1.21}$$

Condition (ii) above ensures that the function $u(x)$ is defined everywhere in Ω and is finite. Let us show that for this synthesized function $u(x)$ the sets F_λ are the cumulative level sets:

$$F_\lambda = \{x \in \Omega :\ u(x) \leq \lambda\}. \tag{1.22}$$

First, if $u(x) \leq \lambda$ then

$$\inf\{\mu :\ x \in F_\mu\} \leq \lambda,$$

hence there exists some $\mu \leq \lambda$ so that $x \in F_\mu$, in which case property (i) above implies that $x \in F_\lambda$. This shows that

$$\{x \in \Omega :\ u(x) \leq \lambda\} \subseteq F_\lambda.$$

On the other hand, if $x \in F_\lambda$, the definition of $u(x)$ implies that $u(x) \leq \lambda$, which in turn means that

$$F_\lambda \subseteq \{x \in \Omega :\ u(x) \leq \lambda\}.$$

Therefore, we have the required identity (1.22).

Let us show that $u(x)$ given by (1.21) is the unique function that has the cumulative level sets F_λ. Suppose there exists another function v such that

$$F_\lambda(u) = F_\lambda(v) \text{ for all } \lambda \in \mathbb{R}.$$

This means that for any $x \in \Omega$ and any $\lambda \in \mathbb{R}$ we have $u(x) \leq \lambda$ if and only if $v(x) \leq \lambda$. Then if at some point $x \in \Omega$ we have $u(x) < v(x)$, we would take $\lambda = (u(x) + v(x))/2$ to reach a contradiction.

Hence, the cumulative level set representation of a function $u(x)$ gives a lossless representation of $u(x)$.

1.4 The Mumford-Shah Free Boundary Image Model

The Mumford-Shah free boundary image model works well for images that represent various objects under the assumption that each object has nearly uniform properties, and the main goal is to capture correctly the interfaces between the objects while the texture does not vary much inside each object. In this case the image can be approximated by piece-wise constant or piece-wise smooth functions, and that is what the Mumford-Shah model does.

1.4.1 Piecewise constant 1D images

Let us first look at what happens in one dimension. Let $u(x)$ be a function defined on the real line $\mathbb{R}$ with

$$\lim_{x\to\pm\infty} u(x) = 0.$$

We say that $u(x)$ is a piecewise constant function if for any point x_0 there exists an interval $(x_0-\varepsilon, x_0+\varepsilon)$ on which $u(x)$ can be written as

$$u(x) = a + bH(x-x_0).$$

Here a and b are two constants (that depend on x_0), and $H(x)$ is the Heaviside function:

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

The constant b is the jump of u at x_0:

$$b = [u]_{x_0} = u(x_0^+) - u(x_0^-).$$

We will denote by

$$J = J_u = \{x : \ [u]_x \neq 0\}$$

the jump set of u.

Theorem 1.9 *The jump set of any piecewise constant function u is closed and isolated.*

Therefore, J can be enumerated as $J = \{x_k\}$ with $k \in S$, where S is a (finite or infinite) subset of $\mathbb{Z}$ (in particular, S can coincide with $\mathbb{Z}$ if x_n extends both to positive and negative infinity), and

$$\ldots < x_n < x_{n+1} < \ldots$$

Then the function $u(x)$ can be written as

$$u(x) = \sum_{n\in S} c_n \chi_{[x_n, x_{n+1})}(x).$$

The gradient (derivative as we are in 1D) is a sum of a bunch of delta functions:

$$\nabla u(x) = \sum_{k\in S} b_k \delta(x - x_k).$$

Here $b_k = c_k - c_{k-1}$ are the jumps at x_k. Therefore, a simple way to encode a piecewise constant function is as follows: we can simply record the locations and values of the jumps (x_n, b_n). With this information we can recover the function $u(x)$ as

$$u(x) = u(0) + \int_0^x \nabla u = u(0) + \sum_{x_n \in (0,x]} b_n. \tag{1.23}$$

The unknown constant $u(0)$ is easy to find from the condition $u(\pm\infty) = 0$:

$$u(0) = -\lim_{K \to +\infty} \sum_{x_n \in (0,K]} b_n.$$

Therefore, (1.23) gives a lossless synthesis of the piece-wise constant function $u(x)$ from the data (x_n, b_n).

1.4.2 Piecewise smooth 1D images

We now generalize the above idea to piecewise smooth images, still in one dimension. We say that a compactly supported function $u(x)$ is piece-wise smooth if for each $x_0 \in \mathbb{R}$ there exists an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ where we can write

$$u(x) = a(x) + b(x)H(x - x_0),$$

with smooth functions $a(x)$ and $b(x)$ – once again, the choice of the functions depends on the point x_0. There is also not a unique way to choose the functions $a(x)$ and $b(x)$ even when x_0 is fixed. What is fixed are the values $b(x_0)$ and $b'(x_0)$: they are given by

$$b(x_0) = [u]_{x_0} = u(x_0^+) - u(x_0^-),$$

and

$$b'(x_0) = [u']_{x_0} = u'(x_0^+) - u'(x_0^-).$$

As before, the jump set of u is the set $J_u = \{x : \ [u]_x \neq 0\}$. Since the function $u(x)$ has compact support, J_u is a finite set:

$$J_u \{x_1 < x_2 < \ldots < x_N\}.$$

We also set $x_0 = -\infty$ and $x_{N+1} = +\infty$ for convenience. Let us denote the derivative $g_n(x) = u'(x)$ on the interval (x_n, x_{n+1}). Then a convenient way to encode the function $u(x)$ is by the following data:

$$g_0(x) \quad \text{and} \quad (x_n, b_n, g_n(x)), \quad 1 \leq n \leq N.$$

Note that compared to the piece-wise constant images we need to record $g_n'(x)$ in addition to (x_n, b_n), in order to hope for a lossless recovery. Such representation is efficient if each $g_n(x)$ is not too complicated and can be well approximated by, say, a lower order polynomial – that is, when $u(x)$ itself is not too oscillatory. The function $u(x)$ can be recovered from this data as

$$u(x) = \sum_{x_n \in (-\infty,x]} b_n + \int_{-\infty}^x \sum_{n=0}^N g_n(y)\chi_{(x_n,x_{n+1})}(y)dy.$$

Another interesting data to encode a piecewise smooth function uses the second derivatives – this also gives the first instance when we see the use of differential equations in imaging. The Poisson representation records the following data:

$$f_0(x) \text{ and } (x_n, u_n^+, u_n^-, f_n(x)), \quad 1 \le n \le N.$$

Here we have denoted $f_n(x) = u''(x)$ on (x_n, x_{n+1}). This representation is particularly efficient when $u(x)$ is nearly linear on each interval (x_n, x_{n+1}) as the functions $f_n(x)$ are all very small then and easy to encode by very few bits.

In order to recover $u(x)$ losslessly on the interval (x_n, x_{n+1}) we solve the second order Poisson equation:

$$u''(x) = f_n(x), \quad x_n < x < x_{n+1}, \tag{1.24}$$

with the boundary data

$$u(x_n^+) = u_n^+, \quad u(x_{n+1}^-) = u_{n+1}^-. \tag{1.25}$$

This problem has a unique solution that allows a lossless synthesis of $u(x)$ from the Poisson data. It is worthwhile to look at the explicit solution of (1.24)-(1.25). It can be written as

$$u(x) = l(x) + w(x), \tag{1.26}$$

with the linear function

$$l(x) = u_n^+ + \frac{u_{n+1}^- - u_n^+}{x_{n+1} - x_n}(x - x_n),$$

and the function $w(x)$ that solves the homogeneous problem

$$w''(x) = f_n(x), \quad x_n < x < x_{n+1}, \tag{1.27}$$

with the boundary data

$$w(x_n^+) = 0, \quad w(x_{n+1}^-) = 0. \tag{1.28}$$

To find $w(x)$ we decompose in the sine series

$$f_n(x) = \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{x_{n+1} - x_n}} \sin\left(\pi k \frac{x - x_n}{x_{n+1} - x_n}\right),$$

then the sine series for $w(x)$ is

$$w(x) = \sum_{k=1}^{\infty} w_k \sqrt{\frac{2}{x_{n+1} - x_n}} \sin\left(\pi k \frac{x - x_n}{x_{n+1} - x_n}\right),$$

with

$$w_k = \frac{a_k}{\pi^2 k^2}.$$

Therefore, if f_k is regular meaning that a_k decay reasonably fast, the coefficients w_k are much smaller than a_k for large k. therefore, in order to represent $w(x)$ reasonably accurately we need to record relatively few coefficients of $f_n(x)$ – this is one of the advantages of the Poisson representation.

1.4.3 Piecewise smooth 2D images

The approach we used in one dimension to descrive piece-wise continuous functions is more difficult to realize in 2D since the jump sets may have a very complicated geometry. Let Ω be a bounded open domain with a Lipschitz boundary. A Lipschitz partition of Ω is a partition of Ω as a finite union

$$\Omega = \bigcup_{n=1}^{N} \Omega_n \bigcup \Gamma. \tag{1.29}$$

Here each Ω_n is a connected open Lipschitz domain (an object in the image), and Γ is a closed subset of Ω that has a finite one-dimensional Hausdorff measure $\mathcal{H}^1(\Gamma) < \infty$.

Theorem 1.10 *Consider a Lipschitz partition of Ω as above. Then (i) $\partial\Gamma = \Gamma$, (ii) $\partial\Omega_n \subseteq \Gamma$ for all $n = 1, \ldots, N$, and*

$$(iii) \qquad \Gamma = \bigcup_{n=1}^{N} \partial\Omega_n.$$

The proof is, once again, simple: (i) follows from the fact that Γ has no interior – the sets in $\mathbb{R}^2$ that have interior ponts can not have finite one-dimensional Hausodorff measure. The second property holds since Ω_n are open and pairwise disjoint. Given (ii), to show (iii) we only need to verify that

$$\Gamma \subseteq \bigcup_{n=1}^{N} \partial\Omega_n. \tag{1.30}$$

Assume that $z \in \Gamma$. Then (i) implies that $z \in \partial\Gamma$, hence there exists a sequence z_k that converges to z such that each $z_k \notin \Gamma$. As there are only finitely many sets Ω_k there has to be a set Ω_m such that infinitely many z_k belong to Ω_m. This means that $z \in \partial\Omega_m$, proving (1.30).

We will then say that a function $u(x)$ is piecewise smooth if there exists a Lipschitz partition of Ω and N smooth functions $U_k(x)$, $k = 1, \ldots, N$ defined in Ω_k such that

$$u(x) = \sum_{k=1}^{N} U_k(x)\chi_k(x),$$

where $\chi_k(x) = \chi_{\Omega_k}(x)$.

Following what we did in one dimension, we may build a Poisson representation of a piecewise smooth function as follows. Set

$$f_n(x) = \Delta U_n(x) \text{ for } x \in \Omega_n,$$

in each sub-domain Ω_n. As each function $U_n(x)$ is smooth inside Ω_n, its trace $\phi_n(x) = U_n(x)$ for $x \in \gamma_n = \partial\Omega_n$ is well defined – this is the limit value of $u(x)$ as it approaches the boundary $\partial\Omega_n$ from inside Ω_n. The Poisson representation for a piece-wise smooth function $u(x)$ then consists of the following data:

$$\{(\gamma_n, \phi_n(x), f_n(x)) : \ 1 \le n \le N\}. \tag{1.31}$$

The functions $U_n(x)$ are then recovered by solving the Poisson equation inside each Ω_n:

$$\begin{aligned} \Delta U_n &= f_n \text{ for } x \in \Omega_n, \\ U_n &= \phi_n \text{ for } x \in \partial\Omega_n. \end{aligned} \tag{1.32}$$

The advantage of this representation is that solving the Poisson equation regularizes the noise in f_n. Indeed, let us consider one domain Ω_k with k fixed and let ψ_m, $m = 1, 2, \ldots$, be the basis of the eigenfunctions for the Dirichlet Laplacian in Ω_k:

$$\begin{aligned} -\Delta\psi_m &= \lambda_m\psi_m \text{ for } x \in \Omega_k, \\ \psi_m &= 0 \text{ for } x \in \partial\Omega_k, \end{aligned} \tag{1.33}$$

normalized so that $\|\psi_m\|_{L^2} = 1$. Assume for simplicity that the boundary data $\phi_k = 0$. The functions ψ_m form a basis for $L^2(\Omega_k)$, hence f_k can be decomposed as

$$f_k(x) = \sum_{m=1}^{\infty} c_m\psi_m(x),$$

with

$$c_m = \int_{\Omega_k} f_k(x)\psi_m(x)dx.$$

The solution of (1.32) is then given by

$$U_n = -\sum_m c_m \frac{f_m}{\lambda_m}\psi_m(x). \tag{1.34}$$

The point is that if the data f_n is noisy then the noisy has mostly components in large m – and these are divided by λ_m, and $\lambda_m \to +\infty$ as $m \to +\infty$. Therefore, a small noise in f_n translates into a smaller noise for U_n – the advantage of the Poisson representation is therefore that it is much less sensitive to noise than a "straight" recording of U_n.

1.4.4 The Mumford-Shah model

In practice, given an image it is quite a non-trivial task to decide where the boundary between various objects (domains Ω_n in the decomposition described in the previous section) is – this is the basic image segmentation problem. An additional difficulty is dealing with the noise and blurring effects that are present in any image. We now describe the Mumford-Shah model that we will encounter also later, that offers one way to deal with these issues.

The recording step in an imaging process can be split into the following steps:

3D scene $\Rightarrow$ an ideal 2D image $u \Rightarrow$ blur $K \Rightarrow$ add noise $n \Rightarrow$ an observed image u_0.

Therefore, the "ideal image" u and the recording u_0 are related by

$$u_0 = K[u] + n.$$

Even if the original image u is piecewise smooth, the recording u_0 is not, due to blurring and noise. The goal of image segmentation is to extract the information about the boundaries of

the objects from the blurred noisy image u_0. In the Mumford-Shah model this is done via a minimization problem. Let us explain how the cost functional is constructed. Let us assume that the original image u came from a domain decomposition

$$\Omega = \bigcup_{m=1}^{M} \Omega_m \bigcup \Gamma,$$

and that the restrictions $u_m = u|_{\Omega_m}$ are smooth functions. The energy of u is defined as

$$E(u, \Gamma) = E_1(\Gamma) + E_2(u|\Gamma).$$

It is natural to take the "edge energy" $E_1(\Gamma)$ to be the length of the interface (or its one-dimensional Hausdorff measure):

$$E_1(\Gamma) = \mathcal{H}^1(\Gamma) = |\Gamma|.$$

The simplest energy of the patches u_m is their Sobolev H^1-norm – it penalizes "unnecessary fluctuations" (a good image is simple):

$$E_2(u|\Gamma) = \sum_{m=1}^{M} \int_{\Omega_m} |\nabla u_m|^2 dx.$$

The combined energy is, therefore:

$$E(u, \Gamma) = \alpha \mathcal{H}^1(\Gamma) + \beta \sum_{m=1}^{M} \int_{\Omega_m} |\nabla u_m|^2 dx. \tag{1.35}$$

The two weights α and β can be chosen according to our preference – whether we penalize more "rough edges" or "rough texture".

Let us assume that the noise n is homogeneous Gaussian white noise. Then its variance can be estimated (if u is given) empirically as

$$\sigma^2 = \frac{1}{|\Omega|} \int_{\Omega} (u_0 - K(u))^2 dx.$$

The Mumford-Shah minimization problem is then to minimize both the energy of the image and the noise variance: minimize

$$E(u, \Gamma|u_0) = \alpha \mathcal{H}^1(\Gamma) + \beta \sum_{m=1}^{M} \int_{\Omega_m} |\nabla u_m|^2 dx + \lambda \int_{\Omega} (u_0 - K(u))^2 dx. \tag{1.36}$$

We assume here that we know the blur operator K. Once again, we can choose the weights α, β and λ according to the nature of the problem.

To summarize: the Mumford-Shah model says that given a recorded blurred noisy image $u_0(x)$ we recover the "best candidate" for the original image is the function $u(x)$ that minimizes the total energy (1.36).

1.4.5 Special BV images

One difficulty in Mumford-Shah model is minimizing over the set of edges Γ – the collection of all edge sets is not a very nice space to use, it has no obvious structure that would allow to use common minimization tools. Essentially, the question is: given a general BV function $u(x)$ – what is the set of its edges?

A natural candidate for such space is the subspace of $\mathrm{BV}(\Omega)$ that consists of special BV functions called $\mathrm{SBV}(\Omega)$ that we will now describe. Let u be a candidate in the Mumford-Shah model with an edge set Γ, which we assume to be piecewise C^1, and let $\mathcal{H}^1$ be the 1D Hausdorff measure on Γ. For almost every point $x \in \Gamma$ we can define the normal ν_x to Γ and the corresponding jump:

$$[u]_x = \lim_{\varepsilon \to 0} (u(x + \varepsilon \nu_x) - u(x - \varepsilon \nu_x)).$$

This allows us to define a vector-valued measure

$$J_u = [u]_x \nu_x d\mathcal{H}^1.$$

It is important to note that it is irrelevant for J_u if we switch the direction of the normal – then both ν and $[u]_x$ switch sign! Then for a Borel subset $\gamma \subseteq \Gamma$ we set

$$J_u(\gamma) = \int_\gamma [u]_x \nu_x d\mathcal{H}^1.$$

Let now $u \in \mathrm{BV}(\Omega)$ and let Du be the (distributional) derivative of u. Using the Radon-Nikodym theorem we can decompose it as

$$Du = \nabla u + D_s u,$$

where ∇u is a measure absolutely continuous with respect to the Lebesgue measure: $\nabla u = f dx$, with $f \in L^1$, and $D_s u$ is singular with respect to the Lebesgue measure. Ideally, we would like to say that $D_s u = J_u$ but things are not that simple. The singular part can be further decomposed as

$$D_s u = D_1 u + D_c u,$$

where, roughly, $D_1 u$ is a measure that has one-dimensional support, and $D_c u$ is supported on a set with the Hausdorff dimension strictly between $d = 1$ and $d = 2$.

Let us now explain how one can find the component $D_1 u$ for a general BV function u. Given a point $x \in \Omega$, a radius $\rho > 0$ and $\lambda \in \mathbb{R}$, we set

$$\{u > \lambda\}_{x,\rho} = \{y \in \Omega \cap B_{z,\rho} : \ u(y) > \lambda\}.$$

We say that u is approximately no greater than λ at x if

$$\lim_{\rho \downarrow 0} \frac{|\{u > \lambda\}_{x,\rho}|}{|B_{x,\rho}|} = 0, \tag{1.37}$$

and write $u \lesssim_x \lambda$. Intuitively this means that if we blow-up a neighborhood around x we will see fewer and fewer points where u takes values larger than λ. We also define

$$u_+(x) = \inf\{\lambda : \ u \lesssim_x \lambda\}, \tag{1.38}$$

and

$$u_-(x) = \sup\{\lambda : \ u \gtrsim_x \lambda\}, \tag{1.39}$$

It is easy to verify that $u_-(x) \leq u_+(x)$. The jump set of a BV(Ω) function u is

$$S_u = \{x \in \Omega : \ u_+(x) > u_-(x)\}. \tag{1.40}$$

Intuitively, S_u is where u has jumps. For instance, for the Heaviside function $H(x)$ we have $H_+(0) = 1$, and $H_-(0) = 0$. Therefore, the set S_u is a natural candidate for the set Γ in the Mumford-Shah model. A remarkable general result is that for any BV(Ω) function u the set S_u can be decomposed as a union of (possible countably many) rectifiable curves and a set S'_u that has the Hausdorff dimension less than one:

$$S_u = S'_u \bigcup_{k=1}^{\infty} \gamma_k, \tag{1.41}$$

where each γ_k is a compact subset of a rectifiable curve. Therefore, at each point of γ_k we can define the normal ν and the jump $[u] = u_+ - u_-$. It can also be shown that

$$D_1 u = [u]\nu d\mathcal{H}^1 = J_u$$

is supported on S_u. Therefore, the vector Radon measure Du of an arbitrary BV function u can be decomposed as

$$Du = \nabla u + J_u + D_c u.$$

We say that a function in BV(Ω) is special if its Cantor component $D_c u$ of the gradient vanishes everywhere. This subspace of BV(Ω) is denoted SBV(Ω).

Replacing the edge set Γ by the jump set S_u we get the weak Mumford-Shah functional that is defined for all functions $u \in$ SBV(Ω): given an image u_0 we set

$$E(u) = \alpha\mathcal{H}^1(S_u) + \beta \int_\Omega |\nabla u|^2 dx + \gamma \int_\Omega (u_0 - K[u])^2 dx. \tag{1.42}$$

A remarkable result is that a solution of this weak energy minimizing problem exists and that $(u, \Gamma = \bar{S}_u)$ is the solution of the original strong formulation. A nice reference for this compactness result as well as other basic facts about SBV functions is the elegant short paper by Alberti and Mantegazza[2] that also has references to the original papers by Ambrosio and DiGiorgi.

2 Introduction to denoising and filtering techniques

2.1 Diffusive filtering

Let us first consider a very simple filtering algorithm on a discrete lattice $\mathbb{Z}^2$: given a discrete image u_{ij} we filter it to get a new image

$$\tilde{u}_{ij} = u_{ij} + \varepsilon \sum_{(kl)\sim(ij)} h_{ij,kl} u_{kl}. \tag{2.1}$$

[2]G. Alberti and C. Mantegazza, A note on the theory of SBV functions, Boll. Un. Mat. Ital. Sezione B, **11**, 1997, 375–382.

Here the notation $(ij) \sim (kl)$ means that the vertices (ij) and (kl) are "neighbors" on the lattice (which in the simplest case can be literally neighbors in $\mathbb{Z}^2$ but can be more general). In order for the filter to do "local averaging" and conserve the total mass

$$M = \sum_{i,j} u_{ij},$$

so that

$$\sum_{i,j} u_{ij} = \sum_{ij} \tilde{u}_{ij},$$

we need to have

$$\sum_{(kl)\sim(ij)} h_{ijkl} = 0, \tag{2.2}$$

for all vertices $(ij) \in \mathbb{Z}^2$. The most canonical example of such filter is taking the 5-point stencil when we set

$$(kl) \sim (ij) \text{ if and only if } |i-k| + |l-j| \leq 1,$$

and

$$h_{ij,kl} = 1 \text{ if } |i-k| + |j-l| = 1,$$

and $h_{ij,ij} = -4$. The filter is then, explicitly:

$$\tilde{u}_{ij} = u_{ij} + \varepsilon[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}]. \tag{2.3}$$

Thinking of the lattice step as $\Delta x = \Delta y$ and filtering as a time-stepping evolution with a time step Δt so that

$$u_{ij} = u(t, i\Delta x, j\Delta y),$$

and

$$\tilde{u}_{ij} = (t + \Delta t, i\Delta x, j\Delta y),$$

we can take $\varepsilon = D\Delta t/(\Delta x)^2$, with a fixed constant D, and re-write (2.3) as

$$\frac{\tilde{u}_{ij} - u_{ij}}{\Delta t} = D\left(\frac{u_{i+1,j} + u_{i-1,j} - 2u_{ij}}{(\Delta x)^2} + \frac{u_{i,j+1} + u_{i,j+1} - 2u_{ij}}{(\Delta y)^2}\right). \tag{2.4}$$

Passing to the limit $\Delta x = \Delta y \to 0$, $\Delta t \to 0$ we obtain the heat equation

$$u_t = D\Delta u, \tag{2.5}$$

with the initial data $u(0, x, y) = u_0(x, y)$ which is the original unfiltered image. The heat equation is posed in a bounded domain Ω, which is our image window, and needs to be supplemented with the boundary conditions on the boundary $\partial\Omega$. A natural choice are the Neumann boundary conditions

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

so as to preserve the total mass

$$M(t) = \int_\Omega u(t, x).$$

Indeed, with these boundary conditions we have

$$\frac{dM}{dt} = \int_\Omega \frac{\partial u}{\partial t} dx = \int_\Omega \Delta u(t,x) dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS = 0.$$

However, the evolution according to the heat equation with the Neumann boundary conditions leads to a uniform final state:

$$u(t,x) \to \frac{1}{|\Omega|} \int_\Omega u_0(x) dx \text{ as } t \to +\infty.$$

This issue can not be fixed by the choice of boundary conditions: for instance, if we impose the Dirichlet Boundary condition

$$u(t,x) = 0 \text{for } x \in \partial\Omega,$$

then in the long time limit we get

$$u(t,x) \to 0 \text{ as } t \to +\infty,$$

which is not much better. Therefore, if we let the heat equation run for too long time, the evolved image will lose all its features: it will be uniformly grey which is usually very far from what you want. One way around this problem is to impose a stopping time, stopping the evolution before the image becomes completely smeared out but after running it for a sufficiently long time to remove the noise. The optimal choice of the stopping time depends on the particular problem. The basic idea of choosing the stopping time can be seen from the Fourier transform of the solution of the heat equation in the whole space: if (2.5) is posed in the whole space, the Fourier transform

$$\hat{u}(t,k) = \int_{\mathbb{R}^n} e^{-ik\cdot x} u(t,x) dx$$

of the solution is

$$\hat{u}(t,k) = e^{-D|k|^2 t} \hat{u}_0(k). \tag{2.6}$$

Therefore, roughly, if we treat as noise oscillations on the scales below $\lambda_0 = 2\pi/k_0$, we should run the diffusion evolution for the time of the order $T_0 \sim \lambda_0^2/D$ so as to remove the oscillations on scales much smaller than λ_0 but keeping the variations on scales much larger than λ_0.

However, if the image $u_0(x)$ has a sharp interface, the linear diffusion equation will smooth it to the same extent as a random noise. For instance, if we consider the heat equation on the interval $-1 \le x \le 1$ with the Neumann boundary conditions:

$$\begin{aligned} &\frac{\partial u}{\partial t} = u_{xx}, \quad -1 \le x \le 1 \\ &u_x(t,-1) = u_x(t,1) = 0, \end{aligned} \tag{2.7}$$

with the initial data $u_0(x) = 0$ for $-1 \le x \le 0$ and $u_0(x) = 1$ for $0 < x \le 1$ then

$$u(t,x) \to 1/2, \text{ as } t \to +\infty, \text{ for all } x \in [-1,1],$$

that is, in the limit $t \to +\infty$ the initial interface is completely removed. This is certainly not what we usually need – we need to devise a filter that removes the noise but keeps the large scale features even if they have sharp interfaces. Therefore, one needs to modify the simple diffusive filtering idea so that at least the sharp interfaces would not be destroyed. One approach to address this issue is by means of a nonlinear diffusion. Once again, we start with a discrete filter, known as the median filter:

$$\tilde{u}_{ij} = \text{median}[u_{kl} : \ (kl) \sim (ij)]. \tag{2.8}$$

In order to understand how this filter works, let us consider a linear function with a jump: $u_n = 0$ for $n > 0$, and

$$u_0 = 1, \ u_{-1} = 0.9, \ u_{-2} = 0.8, \ u_{-3} = 0.7, \ u_{-4} = 0.6, \ u_{-5} = 0.5, \ \text{etc.}$$

If we take the stensil $i \sim i-1, i, i+1$, then the outcome of the median filter is $\tilde{u}_n = 0$ for $n > 0$, and

$$\tilde{u}_0 = 0.9, \ \tilde{u}_{-1} = 0.9, \ \tilde{u}_{-2} = 0.8, \ \tilde{u}_{-3} = 0.7, \ \tilde{u}_{-4} = 0.6, \ \tilde{u}_{-5} = 0.5, \ \text{etc.}$$

so that the jump is well preserved – though very long iterations will kill it eventually.

Similarly, in two dimensions, in the continuous case, let us take

$$u(x, y) = H(ax + by + c),$$

that is, $u(x, y) = 0$ if $ax + by + c < 0$, and $u(x, y) = 1$ for $ax + by + c \geq 0$. Then any radially symmetric median filter will give $\tilde{u}(x, y) = 1$ if $ax + by + c > 0$ and $\tilde{u}(x, y) = 0$ if $ax + by + c < 0$ – therefore, the edge is well preserved also in two dimensions.

Let us see how we may represent filters that preserve edges in the continuous case by a solution of a nonlinear diffusion equation. Consider a filter

$$\tilde{u}_{ij} = \sum_{(kl)\sim(ij)} h_{ij,kl} u_{kl}, \tag{2.9}$$

but we will now allow the weights $h_{ij,kl}$ to depend on u. For instance, consider

$$h_{ij,kl} = \frac{1}{Z_{kl}} \exp\left(-\frac{|u_{ij} - u_{kl}|^\alpha}{\varepsilon}\right). \tag{2.10}$$

Here $\varepsilon \in (0, 1)$ is a parameter that controls how close u_{ij} and u_{kl} have to be for u_{kl} to contribute to $\tilde{u}_{ij}$. If $\varepsilon \ll 1$ is small then u_{kl} does not affect u_{ij} if there is a big difference between u_{ij} and u_{kl} – this is good for preventing destruction of interfaces. On the other hand, if $|u_{ij} - u_{kl}|^\alpha \sim \varepsilon$ then this filter will try to equilibrate the values at u_{ij} and u_{kl}. The power $\alpha > 0$ can also be tuned. Finally, the normalization constant Z_{kl} is chosen so that

$$\sum_{(ij)\sim(kl)} h_{ij,kl} = 1 \text{ for any } (k, l). \tag{2.11}$$

Recall that this condition is needed to ensure the conservation of the total mass

$$\sum_{ij} \tilde{u}_{ij} = u_{ij}.$$

In order to obtain a continuous limit equation, let us also impose the condition

$$\sum_{(kl)\sim(ij)} h_{ij,kl} = 1 \text{ for any } (i,j), \tag{2.12}$$

which means that the filter does local averaging. It is often reasonable to assume that $h_{ij,kl} = h_{kl,ij}$, to ensure (2.12), in the combination with (2.11). Then the filter can be re-written as

$$\tilde{u}_{ij} - u_{ij} = \sum_{(kl)\sim(ij)} h_{ij,kl} u_{kl} - u_{ij}, \tag{2.13}$$

which, with the help of (2.12) becomes

$$\tilde{u}_{ij} - u_{ij} = \sum_{(kl)\sim(ij)} h_{ij,kl}(u_{kl} - u_{ij}). \tag{2.14}$$

We consider the standard neighboring relation in two dimensions: $(ij) \sim (kl)$ if

$$|i-k| + |j-l| \leq 1,$$

as before, and also the following form for $h_{ij,kl}$:

$$h_{ij,kl} = P\left(u_{\frac{i+k}{2},\frac{j+l}{2}}, \nabla u_{\frac{i+k}{2},\frac{j+l}{2}}\right), \tag{2.15}$$

with a smooth function $u(x,y)$. This gives, with $\delta = \Delta x = \Delta y$:

$$\begin{aligned}
\tilde{u}_{ij} - u_{ij} &= h_{ij;i+1,j}(u_{i+1,j} - u_{ij}) + h_{ij;i-1,j}(u_{i-1,j} - u_{ij}) \\
&+ h_{ij;i,j+1}(u_{i,j+1} - u_{ij}) + h_{ij;i,j-1}(u_{i,j-1} - u_{ij}) \\
&= P_{i+\frac{1}{2},j}(u_{i+1,j} - u_{ij}) - P_{i-\frac{1}{2},j}(u_{i,j} - u_{i-1,j}) \\
&+ P_{i,j+\frac{1}{2}}(u_{i,j+1} - u_{ij}) - P_{i,j-\frac{1}{2}}(u_{i,j} - u_{i,j-1}) \\
&\approx \delta[(Pu_x)_{i+\frac{1}{2},j} - (Pu_x)_{i-\frac{1}{2},j} + (Pu_y)_{i,j+\frac{1}{2}} - (Pu_y)_{i,j-\frac{1}{2}}] \approx \delta^2[(Pu_x)_x + (Pu_y)_y].
\end{aligned}$$

Let us take a time step $\Delta t = \varepsilon$, that is, $u_{ij} = u(t, i\delta, j\delta)$ and $\tilde{u}_{ij} = u(t+\varepsilon, i\delta, j\delta)$:

$$\varepsilon u_t \approx \delta^2[(Pu_x)_x + (Pu_y)_y]. \tag{2.16}$$

Assuming $\varepsilon = \delta^2$ we get a nonlinear diffusion equation

$$u_t = \nabla \cdot [P(u, \nabla u)\nabla u]. \tag{2.17}$$

A difficult question to which we will come back often is which nonlinear diffusion equation should be used for various images so that particular image features would be preserved. Another important mathematical issue is that often the nonlinear PDE that would optimally satisfy our requirements is highly nonlinear so that the existence theory for solutions may be very complicated.

2.2 Denoising methods for BV functions

The difference between L^2 and TV errors

Let us first recall why the total variation means of measuring the error is different from what you obtain with the L^2 error. Let $\bar{x}$ be a scalar to be measured and $x_1, x_2, \ldots, x_n$ be the measurements of x that contain errors. The L^2 estimate for the error is

$$e_2(x) = \sum_{k=1}^{N} (x - x_k)^2,$$

and the best prediction for x that minimizes the L^2-error is

$$\bar{y}_2 = \text{argmin } e_2(x) = \frac{1}{N} \sum_{k=1}^{N} x_k.$$

This prediction is reasonable but is prone to huge errors if N is not too large but one of x_k is very far from $\bar{x}$, due to a measurement error. For instance, if $N = 10$ and all $x_j = 1$ except for $x_{10} = 100$, we get

$$\bar{y}_2 = \frac{109}{10},$$

which is very far from the true value $\bar{x} = 1$.

Consider what happens if, instead of the L^2 error we use the TV error

$$e_1(x) = \sum_{k=1}^{N} |x - x_k|.$$

Then the best TV estimate for $\bar{x}$ is (this is a nice calculus exercise)

$$\bar{x}_1 = \text{median}(x_j).$$

Therefore, in our previous example we would have $\bar{x}_1 = 1$, so that one fluke measurement would not throw the result as far off as in the L^2 estimator.

Biased iterated median filtering

Before we describe the Rudin, Osher, Fatemi denoising model based on TV minimization, we consider a much simpler filter that will lead us to the full model. Let x_k, $k = 0, 1, \ldots, N$ be a 1D discretized signal on $[0, 1]$. We assume that $x_j = x(j/N)$ for some continuous function $x(t)$ but that the measured signal y_k is noisy:

$$y_k = x_k + n_k,$$

where n_k is noise. In order to remove the noise we start an iterative estimation process based on localized median filtering. Given the iterate $x_k^{(n)}$ let us define the error estimator

$$e_\lambda(z; x_{k-1}^{(n)}, x_{k+1}^{(n)}, y_k) = |z - x_{k-1}^{(n)}| + |z - x_{k+1}^{(n)}| + \lambda |z - y_k|,$$

for $z \in \mathbb{R}$. The parameter λ regulates how much we weigh the bias toward the initial measurement y_k relative to the bias toward the median value that brings about smoothness and kills the noise. We update our estimator as follows:

$$x_k^{(n+1)} = \text{argmin}_z e_\lambda(z; x_{k-1}^{(n)}, x_{k-1}^{(n)}, y_k), \quad k = 0, 1 \ldots, N.$$

When $\lambda = 1$ we simply have

$$x_k^{(n+1)} = \text{median}(x_{k-1}^{(n)}, x_{k+1}^{(n)}, y_k),$$

but for other values of λ we have different expressions (explicit but more cumbersome). We can define the total cost function as

$$\tilde{E}_\lambda(z; x^{(n)}, y) = \sum_{k=1}^{N} e_\lambda(z_k; x_{k-1}^{(n)}, x_{k+1}^{(n)}, y_k),$$

and write the iteration process as

$$x^{(n+1)} = \text{argmin}_z \tilde{E}_\lambda(z; x^{(n)}, y). \tag{2.18}$$

The following theorem shows how this iteration process balances regularity and bias toward the original measurement.

Theorem 2.1 *If the above iteration process converges:*

$$\lim_{n \to +\infty} x_k^{(n)} = \bar{x}_k, \quad k = 1, 1, \ldots, N,$$

then the limit $\bar{x} \in \mathbb{R}^N$ is the critical point of the cost function

$$E_\lambda(z; y) = \sum_{k=0}^{N} |z_{k+1} - z_k| + \lambda \sum_{k=1}^{N} |z_k - y_k|. \tag{2.19}$$

Proof. Equation (2.18) implies that the limit should be a fixed point of the iteration process and $\bar{x}_k$ should satisfy

$$\bar{x}_k = \text{argmin}_t e_\lambda(t; \bar{x}_{k+1}, \bar{x}_{k-1}, y_k) = \text{argmin}_t [|t - \bar{x}_{k+1}| + |t - \bar{x}_{k-1}| + \lambda |t - y_k|], \tag{2.20}$$

for all $k = 0, 1, \ldots, N$. On the other hand, each component $\bar{z}_k$, $k = 1, \ldots, N$ of the minimizer of $E_\lambda(z, ; y)$ should satisfy the minimization problem with all other components $\bar{z}_0, \ldots, \bar{z}_{k-1}, \bar{z}_{k+1}, \ldots, \bar{z}_N$ fixed:

$$\bar{z}_k = \text{argmin}_t [|t - \bar{z}_{k+1}| + |t - \bar{z}_{k-1}| + \lambda |t - y_k|], \tag{2.21}$$

which is exactly the same equation as (2.20). Therefore, the limit $\bar{x}$ and the minimizer $\bar{z}$ of $E_\lambda(z; y)$ have to coincide.

The Rudin-Osher TV denoising model

Let us now consider the continuous version of the cost function $E_\lambda(z;y)$ given by (2.19), first in one dimension. Let $x_0(t)$ be our noisy measurement, and $z(t)$ a guess for the true non-noisy image. We define the cost functional as a continuous version of that in Theorem 2.1:

$$E_\lambda[z(t);x_0(t)] = \int_0^1 |z'(t)|dt + \lambda \int_0^1 |z(t)-x_0(t)|dt. \tag{2.22}$$

A more general form of (2.22) is

$$E_{\lambda,p}[z(t);x(t)] = \int_0^1 |z'(t)|dt + \lambda \int_0^1 |z(t)-x_0(t)|^p dt, \tag{2.23}$$

for some $p \geq 1$. Then the denoising estimator is

$$\bar{x}_{\lambda,p} = \text{argmin}_z E_{\lambda,p}[z;x_0], \tag{2.24}$$

with the minimum taken over an appropriate class of functions.

Generalizing this functional to two dimensions and taking $p = 2$ (which is a particularly reasonable choice, when the noise is a Gaussian white noise), we obtain the Rudin-Osher denoising model:

$$E_\lambda[u;u_0] = \int_\Omega |\nabla u|dx + \frac{\lambda}{2}\int_\Omega |u(x)-u_0(x)|^2 dx, \tag{2.25}$$

where $u_0(x)$ is our noisy measurement. However, the minimizer u may not be attained in the Sobolev space $W^{1,1}(\Omega)$. In order to guarantee that the minimizer exists we should consider the minimization problem over the larger class of $\text{BV}(\Omega)$ functions and modify the functional accordingly:

$$E_{TV}[u;u_0] = \int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega |u(x)-u_0(x)|^2 dx, \tag{2.26}$$

Theorem 2.2 *Assume that $u_0 \in L^2(\Omega)$. Then the minimizer of $E_{TV}(u;u_0)$ exists in $BV(\Omega)$ and is unique.*

Proof. The functional $E_{TV}(u;u_0)$ is strictly convex in u – because of the second term that is quadratic in u (the first term is linear in u and is thus convex but not strictly convex). Therefore, the minimizer (if it exists) is unique. Existence of the minimizer can be shown as follows. There exist $u \in \text{BV}(\Omega)$ such that $E_{TV}(u;u_0)$ is finite – take, for instance, any $u \in L^2(\Omega) \cap \text{BV}(\Omega)$. Next, if u_n is a minimizing sequence such that

$$E_{TV}(u_n;u_0) \to \bar{E} = \min_{u \in \text{BV}_{(\Omega)}} E_{TV}(u;u_0),$$

then the sequence $E_{TV}(u_n;u_0)$ is uniformly bounded, hence the sequence u_n is bounded in $\text{BV}(\Omega)$. As u_n is bounded in $\text{BV}(\Omega)$, Theorem 1.7 implies that there is a subsequence u_{n_k} which converges as $k \to +\infty$ in $L^1(\Omega)$ to a limit $\bar{u}$. Theorem 1.4 implies that $\bar{u} \in \text{BV}(\Omega)$, and, moreover,

$$\|\bar{u}\|_{\text{BV}_{(\Omega)}} \leq \liminf_{k\to+\infty} \|u_{n_k}\|_{\text{BV}_{(\Omega)}}.$$

In addition, convergence in $L^1(\Omega)$ implies that

$$\int_\Omega |\bar{u}(x) - u_0(x)|^2 dx \leq \liminf_{k\to+\infty} \int_\Omega |u_{n_k}(x) - u_0(x)|^2 dx.$$

We conclude that

$$E_{TV}(\bar{u}; u_0) \leq \bar{E} = \liminf_{k\to+\infty} E_{TV}(u_{n_k}; u_0).$$

As a consequence, $\bar{u}$ is a minimizer of $E_{TV}(u; u_0)$ over $\mathrm{BV}(\Omega)$ and we are done.

The Euler-Lagrange equation for the Rudin-Osher model

In order to devise an algorithm to minimize the Rudin-Osher functional

$$E_{TV}[u; u_0] = \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega |u(x) - u_0(x)|^2 dx, \tag{2.27}$$

we will look at its formal Euler-Lagrange equation. Let us consider an infinitesimal perturbation $u + \delta v$, with $\delta \ll 1$. Assuming that u and v are sufficiently smooth, we get

$$\begin{aligned}
&E_{TV}(u + \delta v; u_0) = \int_\Omega |\nabla u + \delta \nabla v| + \frac{\lambda}{2} \int_\Omega |u + \delta v - u_0|^2 dx \\
&\approx E_{TV}(u; u_0) + \delta \int_\Omega \frac{1}{|\nabla u|} \nabla u \cdot \nabla v + \lambda\delta \int_\Omega v(u - u_0) dx \\
&= E_{TV}(u; u_0) - \delta \int_\Omega \nabla \cdot \left(\frac{1}{|\nabla u|} \nabla u \right) v + \delta \int_{\partial\Omega} \frac{1}{|\nabla u|} \frac{\partial u}{\partial n} v d\mathcal{H}^1 + \lambda\delta \int_\Omega v(u - u_0) dx.
\end{aligned}$$

Here n is the unit normal to $\partial\Omega$ and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure along $\partial\Omega$. Hence, if the unique minimizer is smooth (it lives in $W^{1,1}(\Omega)$), it should satisfy the Euler-Lagrange boundary value problem

$$\begin{aligned}
&-\nabla \cdot \left(\frac{1}{|\nabla u|} \nabla u \right) + \lambda(u - u_0) = 0, \text{ in } \Omega \\
&\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega.
\end{aligned} \tag{2.28}$$

This is formally an elliptic equation but it degenerates where $\nabla u = 0$ – if the image is homogeneous in some region the diffusion coefficient there is infinite. This is good since this gives very fast smoothing in the regions where u should be homogeneous. On the other hand, in the regions where u has a very large gradient, the diffusion coefficient vanishes meaning that no smoothing takes place – this is good news for preserving the edges. Alternatively, one could adopt a time evolution

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{1}{|\nabla u|} \nabla u \right) + \lambda(u - u_0) = 0, \text{ in } \Omega \\
&\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega,
\end{aligned} \tag{2.29}$$

with some initial condition, such as $u(0,x) = u_0(x)$. Then one would hope that as $t \to +\infty$ solutions of the degenerate parabolic problem (2.29) would converge to the steady state that solves the degenerate elliptic problem (2.28).

The natural difficulty in trying to solve the elliptic problem (2.28) or the parabolic equation (2.29) is, obviously, in their highly nonlinear nature and the possible singularity of the diffusion coefficient $D = 1/|\nabla u|$ at the critical points of u. One natural regularization of the parabolic equation is to consider instead

$$\begin{aligned} &\frac{\partial u}{\partial t} - |\nabla u| \nabla \cdot \left(\frac{1}{|\nabla u|} \nabla u \right) - \lambda |\nabla u|(u - u_0) = 0, \text{ in } \Omega \\ &\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega. \end{aligned} \tag{2.30}$$

This equation has the same steady states as the original elliptic problem (2.28) but the degeneracy at the places where $|\nabla u| = 0$ is now removed. This approach was proposed by Marquina and Osher and works quite well.

Another regularization works directly for the elliptic problem. The idea is to replace $|\nabla u|$ by

$$|\nabla u|_a = \sqrt{|\nabla u|^2 + a^2}, \tag{2.31}$$

with a small regularization parameter $a \ll 1$. The regularized elliptic problem is

$$\begin{aligned} &-\nabla \cdot \left(\frac{1}{|\nabla u|_a} \nabla u \right) + \lambda(u - u_0) = 0, \text{ in } \Omega \\ &\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega. \end{aligned} \tag{2.32}$$

This is the Euler-Lagrange equation for the regularized cost functional

$$E_a[u; u_0] = \int_\Omega |Du|_a + \frac{\lambda}{2} \int_\Omega |u(x) - u_0(x)|^2 dx. \tag{2.33}$$

The advantage of (2.32) is that there is no longer a problem at the points where $\nabla u(x) = 0$, and the diffusivity is still small around the edges where $|\nabla u|$ is large.

Duality for the TV denoising model

The approach to the TV minimization problem described above is a primal method that directly approaches the minimization problem. We will now describe the dual formulation that involves a min-max problem whose critical points are the same as the minimizer of the original problem. The primal problem is to minimize the TV cost functional

$$E_{TV}[u; u_0] = \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega |u(x) - u_0(x)|^2 dx, \tag{2.34}$$

Let us recall the definition of the total variation:

$$\int_\Omega |\nabla u| = \sup \left[\int_\Omega u(\nabla \cdot g) dx : \ g = (g_1, g_2), \ g_{1,2} \in C^1(\Omega), |g_1|^2 + |g_2|^2 \leq 1 \right]. \tag{2.35}$$

Using this expression in (2.34) gives the problem

$$\min_u \left(\sup_g \int_\Omega u(\nabla \cdot g)dx + \frac{\lambda}{2} \int_\Omega |u(x) - u_0(x)|^2 dx \right), \tag{2.36}$$

which can be re-written as

$$\min_u \sup_g \left(\int_\Omega u(\nabla \cdot g)dx + \frac{\lambda}{2} \int_\Omega |u(x) - u_0(x)|^2 dx \right). \tag{2.37}$$

Here the supremum over g is taken over all admissible g as in the definition (2.35) of the total variation. We will denote the class of these admissible functions by $\mathcal{A}$.

Formally exchanging the min and the sup in (2.37) gives the max-min problem

$$\sup_{g \in \mathcal{A}} \min_u \left(\int_\Omega u(\nabla \cdot g)dx + \frac{\lambda}{2} \int_\Omega |u(x) - u_0(x)|^2 dx \right). \tag{2.38}$$

The minimization problem in u is now trivial – we can simply minimize pointwise the integrand

$$u(\nabla \cdot g) + \frac{\lambda}{2}(u - u_0)^2,$$

which leads to

$$\bar{u} = u_0 - \frac{1}{\lambda} \nabla \cdot g, \tag{2.39}$$

so that

$$\bar{u}(\nabla \cdot g) + \frac{\lambda}{2}(\bar{u} - u_0)^2 = u_0(\nabla \cdot g) - \frac{1}{2\lambda} |\nabla \cdot g|^2.$$

Substituting this back into (2.38) gives the dual formulation

$$\sup_{g \in \mathcal{A}} \int_\Omega \left(u_0(\nabla \cdot g) - \frac{1}{2\lambda} |\nabla \cdot g|^2 \right) dx. \tag{2.40}$$

If we find the critical point $g = (g_1, g_2)$ of (2.40), the image $\bar{u}$ can be recovered via (2.39). The advantage of the dual formulation is that it is smooth in ∇g unlike the original formulation that depends on $|\nabla u|$ and is not smooth. The disadvantage is that we now deal with a constrained optimization problem – the class $\mathcal{A}$ involves the pointwise constraint

$$g_1^2 + g_2^2 \leq 1.$$

In order to deal with this issue, note that if at some point $x \in \Omega$ the maximizer $|g(x)| < 1$ then at this point $g(x)$ should satisfy the Euler-Lagrange equation for the unconstrained problem

$$\frac{1}{\lambda} \nabla(\nabla \cdot g) - \nabla u_0 = 0, \tag{2.41}$$

while if $|g(x)| = 1$ at some $x \in \Omega$ then at such point we have

$$\frac{1}{\lambda} \nabla(\nabla \cdot g) - \nabla u_0 = \mu(x) g, \tag{2.42}$$

with some Lagrange multiplier $\mu(x)$. In both cases we may write

$$\mu(x) = |H(g)|,$$

where

$$H(g) = \frac{1}{\lambda}\nabla(\nabla \cdot g) - \nabla u_0. \tag{2.43}$$

Therefore, both (2.41) and (2.42) can be written simultaneously as

$$H(g) - |H(g)|g = 0, \tag{2.44}$$

with $H(g)$ defined by (2.43). Note that the constraint $|g| \leq 1$ is enforced automatically for the solution of (2.44). Moreover, it follows from (2.44) that $H(g) = 0$ where $|g| < 1$.

One way to solve the nonlinear equation (2.44) numerically is to consider the evolution equation

$$g_t = H(g) - |H(g)|g, \tag{2.45}$$

and hope that the long time limit of the solution of (2.45) is the steady solution of (2.44). One corresponding numerical scheme is the semi-explicit iteration

$$g^{(n+1)} = g^{(n)} + (H(g^{(n)}) - |H(g^{(n)})|g^{(n+1)})\Delta t,$$

that is,

$$g^{(n+1)} = \frac{1}{1 + |H(g^{(n)}|\Delta t}(g^{(n)} + H(g^{(n)})\Delta t).$$

Note that the primal problem for u:

$$-\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(u - u_0) = 0, \tag{2.46}$$

and the dual equation (2.45) for the vector g are related in a simple way. Let g solve the dual problem (2.45), and set

$$u = u_0 - \frac{1}{\lambda}\nabla \cdot g, \tag{2.47}$$

then $H(g) = -\nabla u$, and thus u satisfies

$$\nabla u = |\nabla u|g. \tag{2.48}$$

Dividing by $|\nabla u|$ and taking the divergence gives

$$\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) = \nabla \cdot g,$$

that is, recalling (2.47):

$$-\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(u - u_0) = 0, \tag{2.49}$$

which is (2.46). On the other hand, the system (2.47)-(2.48) gives a primal-dual formulation that involves both u and g as unknowns, and we will come back to it later.

A convenient regularization of this system is

$$\begin{aligned} u &= u_0 - \frac{1}{\lambda}\nabla \cdot g, \\ \nabla u &= |\nabla u|_a g. \end{aligned} \tag{2.50}$$

Here $|\nabla u|_a = \sqrt{|\nabla u|^2 + a^2}$, as in (2.31). The regularized system may be solved iteratively.

Properties of the minimizers of the Rudin-Osher functional

A useful Banach space associated with the Rudin-Osher functional is defined as follows. Consider all distributions f that can be written as

$$f = \nabla \cdot g = \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2}, \tag{2.51}$$

in the sense of distributions, with an L^∞ vector valued function g:

$$\|g\|_{L^\infty} = \sup_{x \in \Omega} \sqrt{g_1^2 + g_2^2} < +\infty.$$

Note that the representation (2.51) is not unique: if $f = \nabla \cdot g$, then $f = \nabla \cdot g'$, for any $g' = g + g_0$ with $\nabla \cdot g_0 = 0$. Given $f \in G$ we set

$$G_f = \{g \in L^\infty : \ f = \nabla \cdot g\}. \tag{2.52}$$

The norm in the space G is defined as

$$\|f\|_* = \inf_{g \in G_f} \|g\|_\infty. \tag{2.53}$$

The infimum in (2.53) is actually attained at some $g \in G_f$. In order to see that, let us assume that $g_n \in G_f$ and

$$\|g_n\|_\infty \to \|f\|_*,$$

as $n \to +\infty$. As g_n is bounded in $L^\infty(\Omega)$ which is the dual space of $L^1(\Omega)$, it follows that there exists a sub-sequence g_{n_k} that converges to a limit $g \in L^\infty$, in the weak-* topology. That is, for any vector-valued function $\phi \in L^1(\Omega)$ we have

$$\int_\Omega (g_{n_k} \cdot \phi) dx \to \int_\Omega (g \cdot \phi) dx, \tag{2.54}$$

as $k \to +\infty$. Moreover, we have

$$\|g\|_\infty \leq \liminf_{k \to +\infty} \|g_{n_k}\|_\infty.$$

We also have, for any test function $\psi \in C_c^\infty(\Omega)$, since all g_{n_k} lie in G_f:

$$\int_\Omega \psi f dx = \int_\Omega \psi (\nabla \cdot g_{n_k}) dx = -\int_\Omega (\nabla \psi \cdot g_{n_k}) dx \to -\int_\Omega (\nabla \psi \cdot g) dx = \int_\Omega \psi (\nabla \cdot g) dx,$$

hence $\nabla \cdot g = f$ and thus $g \in G_f$. We conclude that $g \in G_f$ and $\|f\|_* = \|g\|_\infty$.

Let us check that $\|\cdot\|_*$ is, indeed, a norm. It is obvious that $\|\lambda f\|_* = |\lambda| \|f\|_*$. To check the triangle inequality, notice that if $g_1 \in G_{f_1}$ and $g_2 \in G_{f_2}$ then $g_1 + g_2 \in G_{f_1 + f_2}$. Therefore, if we take $g_{1,2}$ so that $\|f_{1,2}\|_* = \|g_{1,2}\|_\infty$, we see that

$$\|f_1 + f_2\|_* \leq \|g_1\|_\infty + \|g_2\|_\infty = \|f_1\|_* + \|f_2\|_*.$$

Moreover, G is a Banach space. Indeed, let f_n be a Cauchy sequence in G. In order to show that the sequence f_n converges to a limit in G it is sufficient to prove that f_n has a

subsequence f_{n_k} that converges strongly in G. Choose $g_n \in G_{f_n}$ such that $\|f_n\|_* = \|g_n\|_\infty$. Then g_n is a bounded sequence in $L^\infty(\Omega)$, thus there exists a subsequence g_{n_k} that converges to a limit $g \in L^\infty(\Omega)$ in the weak-* topology, that is, (2.54) holds for all $\phi \in L^1(\Omega)$. As $f_n = \nabla \cdot g_n$, it follows that $f_{n_k} \to f = \nabla \cdot g$ in the sense of distributions. We will show that actually f_{n_k} converges to f in the strong topology of G:

$$\|f_{n_k} - f\|_* \to 0.$$

Let us fix $k \geq 1$ and for each $l \geq k$ take $g_{l;k}$ so that

$$\nabla \cdot g_{l;k} = f_{n_l} - f_{n_k},$$

and

$$\|g_{l;k}\|_\infty = \|f_{n_l} - f_{n_k}\|_* = \varepsilon_{l;k}.$$

Note that $\varepsilon_{l;k} \to 0$ as $l \to +\infty$ since the sequence f_n is Cauchy. Define $p_{l;k} = g_{n_k} + g_{l;k}$, then

$$\nabla \cdot p_{l;k} = f_{n_k} + (f_{n_l} - f_{n_k}) = f_{n_l},$$

so that $p_{l;k} \in G_{f_{n_l}}$. The sequence $p_{l;k}$ is bounded in L^∞, and by the same token as before, after possibly extracting a subsequence, $p_{l;k}$ converges in the weak-* topology of $L^\infty(\Omega)$ to a limit $q_k \in L^\infty(\Omega)$. As a consequence, the sequence f_{n_l} converges as $l \to +\infty$ to $\nabla \cdot q_k$ as $l \to +\infty$, in the sense of distributions, for any k fixed. Uniqueness of the limit in the sense of distributions implies that $f = \nabla \cdot q_k$, for all k. We conclude that

$$\|f_{n_k} - f\|_* \leq \|g_{n_k} - q_k\|_\infty \leq \liminf_{l\to\infty} \|g_{n_k} - p_{l;k}\|_\infty = \liminf_{l\to\infty} \|g_{l;k}\|_\infty = \liminf_{l\to\infty} \varepsilon_{l;k} = 0,$$

and we are done. Therefore, the space G is a Banach space.

We have the following observation.

Proposition 2.3 *Let $v \in G \cap L^2(\Omega)$ and $u \in BV(\Omega)$, and assume that u has zero trace on $\partial\Omega$, then*

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|v\|_* \int_\Omega |Du|. \tag{2.55}$$

Proof. If $\phi \in C_c^\infty(\Omega)$ is a smooth test function that vanishes on $\partial\Omega$, then for any $g \in G_v$ we have

$$\int_\Omega \phi(x)v(x)dx = -\int_\Omega (\nabla\phi) \cdot g dx \leq \|g\|_\infty \|\nabla\phi\|_{L^1(\Omega)},$$

and thus

$$\int_\Omega \phi(x)v(x)dx \leq \|v\|_* \|\nabla\phi\|_{L^1(\Omega)}.$$

By density this inequality extends to all $\phi \in W_0^{1,1}(\Omega)$ (the Sobolev space of $W^{1,1}$ functions with zero trace in Ω). Theorem 1.5 says that for any $u \in \mathrm{BV}(\Omega)$ with zero trace we can find a sequence $u_n \in W_0^{1,1}$ such that $u_n \to u$ on $L^1(\Omega)$ and

$$\int_\Omega |\nabla u_n| dx \to \int_\Omega |Du|.$$

It follows from the previous argument that

$$\left|\int_\Omega u_n(x)v(x)dx\right| \leq \|v\|_* \int_\Omega |\nabla u_n| dx,$$

and passing to the limit we get

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \|v\|_* \int_\Omega |Du|,$$

finishing the proof.

Let us now return to the Rudin-Osher minimization problem that we write now as follows: given $f \in L^2(\Omega)$ find

$$\min\left[\int_\Omega |Du| + \lambda \int_\Omega v^2 dx : \quad u \in \mathrm{BV}(\Omega), v \in L^2(\Omega),\ u+v=f\right]. \tag{2.56}$$

Recall that our hope is that if $(\bar{u}, \bar{v})$ minimize this functional, then in the decomposition $f = \bar{u} + \bar{v}$, the function $\bar{u}$ represents "objects' and $\bar{v}$ represents "texture and noise". The next result shows that is not the case if the image f is too small in the G-norm.

Theorem 2.4 *Assume that $\Omega = \mathbb{R}^2$, and that*

$$\lambda \|f\|_* \leq \frac{1}{2}. \tag{2.57}$$

Then the minimizer of (2.56) is $\bar{u} = 0$ and $\bar{v} = f$.

Proof. Assume that $f = u + v$ as in (2.56) then

$$\int_{\mathbb{R}^2} |Du| + \lambda \int_{\mathbb{R}^2} |v|^2 dx = \int_{\mathbb{R}^2} |Du| + \lambda \int_{\mathbb{R}^2} |f-u|^2 dx = \int_{\mathbb{R}^2} |Du| + \lambda \|u\|_2^2 + \lambda \|f\|_2^2 - 2\lambda \langle u, f\rangle.$$

The inequality in Proposition 2.3 implies then

$$\int_{\mathbb{R}^2} |Du| + \lambda \int_{\mathbb{R}^2} |v|^2 dx \geq \int_{\mathbb{R}^2} |Du| + \lambda \|u\|_2^2 + \lambda \|f\|_2^2 - 2\lambda \|f\|_* \int_{\mathbb{R}^2} |Du|.$$

Using the bound (2.57) we get

$$\int_{\mathbb{R}^2} |Du| + \lambda \int_{\mathbb{R}^2} |v|^2 dx \geq \lambda \|u\|_2^2 + \lambda \|f\|_2^2 \geq \lambda \|f\|_2^2,$$

whence the minimizer is $\bar{u} = 0$, $\bar{v} = f$.

Therefore, if f is too small then the Rudin-Osher minimization problem does nothing to denoise the image, or, alternatively, simply kills the image! Let us see what happens when $\lambda \|f\|_* \geq 1/2$.

Theorem 2.5 *Assume that* $\Omega = \mathbb{R}^2$, *and*

$$\lambda \|f\|_* \geq \frac{1}{2}. \tag{2.58}$$

Then the minimizer $(\bar{u}, \bar{v})$ *is characterized by the following two conditions, in addition to* $f = \bar{u} + \bar{v}$*:*

$$\|\bar{v}\|_* = \frac{1}{2\lambda} \text{ and } \langle \bar{u}, \bar{v} \rangle = \|\bar{v}\|_* \int_{\mathbb{R}^2} |D\bar{u}|. \tag{2.59}$$

Moreover, (2.59) implies that $f = \bar{u} + \bar{v}$ *satisfies (2.58) if* $\bar{u} \not\equiv 0$.

This result says that if the image f is sufficiently large so that we do get some objects and not only "texture and noise' then still the level of "texture and noise" is fixed and does not grow if f increases while keeping λ fixed, as one would naturally expect! A word of caution is in order, however: it is the G-norm of v that has a uniform upper bound. As we will see later, the L^2-norm of $\bar{v}$ can be arbitrarily large.

Proof. First, let $(\bar{u}, \bar{v})$ be the minimizer of the Rudin-Osher functional, then for any admissible perturbation h we have

$$\int |D\bar{u}| + \lambda \|\bar{v}\|_2^2 \leq \int |D(\bar{u} + \varepsilon h)| + \lambda \|\bar{v} - \varepsilon h\|_2^2 \leq \int |D\bar{u}| + |\varepsilon| \int |Dh| + \lambda \|\bar{v} - \varepsilon h\|_2^2, \tag{2.60}$$

that is:

$$0 \leq |\varepsilon| \int |Dh| + \lambda \varepsilon^2 \|h\|_2^2 - 2\lambda \varepsilon \langle \bar{v}, h \rangle,$$

whence

$$2\lambda |\langle \bar{v}, h \rangle| \leq \int |Dh|$$

for all admissible perturbations h. It follows that $2\lambda \|\bar{v}\|_* \leq 1$.

On the other hand, taking $h = \bar{u}$ in (2.60) gives

$$2\lambda \varepsilon \langle \bar{v}, \bar{u} \rangle \leq (|1 + \varepsilon| - 1) \int |D\bar{u}| + \varepsilon^2 \lambda^2 \int |\bar{u}|^2 dx.$$

As this should be true both for $\varepsilon > 0$ and $\varepsilon < 0$, we conclude that

$$2\lambda \int \bar{u}(x)\bar{v}(x)dx = \int |D\bar{u}|.$$

As $2\lambda \|\bar{v}\|_* \leq 1$, it follows now that we must also have $\|v\|_* = 1/(2\lambda)$. Therefore, the minimizer satisfies the two conditions in (2.59). On the other hand, assume that $\bar{u} + \bar{v} = f$ and $(\bar{u}, \bar{v})$ satisfies (2.59):

$$\|\bar{v}\|_* = \frac{1}{2\lambda} \text{ and } \langle \bar{u}, \bar{v} \rangle = \|\bar{v}\|_* \int_{\mathbb{R}^2} |D\bar{u}|. \tag{2.61}$$

Then for any ε and function h we have, using the duality inequality and (2.61):

$$\int |D(\bar{u} + \varepsilon h)| + \lambda \int (\bar{v} - \varepsilon h)^2 \geq 2\lambda \int (\bar{u} + \varepsilon h)\bar{v} + \lambda \|\bar{v}\|_2^2 - 2\lambda \varepsilon \int \bar{v}(x)h(x)dx + \lambda \varepsilon^2 \|h\|_2^2$$

$$= 2\lambda \int \bar{u}(x)\bar{v}(x)dx + \lambda \|\bar{v}\|_2^2 + \lambda \varepsilon^2 \|h\|_2^2 = \int |D\bar{u}| + \lambda \|\bar{v}|_2^2 + \lambda \varepsilon^2 \|h\|_2^2 \geq \int |D\bar{u}| + \lambda \|\bar{v}\|_2^2,$$

so that $(\bar{u}, \bar{v})$ is the Rudin-Osher minimizer. As a consequence, if $\bar{u} \not\equiv 0$ we must have $\lambda \|f\|_* \geq 1/2$.

Let us now understand some consequences of the previous two theorems. Think about denoising as the map $\Phi : \ f \to u$, where u is the BV component of the minimizer of the Rudin-Osher functional. In a loose sense, it does a version of soft thresholding: we represent $f = \bar{u} + \bar{v}$, with $\bar{u} = \Phi(f)$, and if $\|f\|_* \leq 1/(2\lambda)$, then $\Phi(f) = 0$, while if $\|f\|_* \geq 1/(2\lambda)$, to get $\Phi(f)$, we subtract from f a function $\bar{v}$ with $\|\bar{v}\|_* = 1/(2\lambda)$.

The map Φ does not always work quite as we would think intuitively. For example, if we have a simple image of the type $f(x) = \chi_\Omega(x)$ where Ω is a bounded domain with a smooth boundary, we might expect that such image would be preserved by the Rudin-Osher functional. Sadly, Theorem 2.5 says that this is not the case – the "noise+texture" component v has a fixed G-norm: $\|v\|_* = 1/(2\lambda)$ and thus $\bar{u} \neq f$. On the other hand, things are not too bad as can be seen from the next example.

Theorem 2.6 *Let $f(x) = \alpha \chi_{B_R}(x)$, where B_R is a disk of radius R centered at the origin and $\alpha > 0$. Then its Rudin-Osher decomposition is $f = \bar{u} + \bar{v}$, where*

$$\bar{u} = (\alpha - \frac{1}{\lambda R})\chi_{B_R}(x), \quad \bar{v} = \frac{1}{\lambda R}\chi_{B_R}(x), \tag{2.62}$$

if $\lambda R \geq 1/\alpha$, while

$$\bar{u} = 0, \quad \bar{v} = f = \alpha \chi_{B_R}(x),$$

if $\lambda R < 1/\alpha$.

There are two remarkable observations here; first, if R is sufficiently large then the location of the edges is recovered perfectly – this is excellent news. Second, no matter how large α is, the "noise+texture" component $\bar{v}$ does not depend on α if $\lambda R \geq 1/\alpha$.

Proof. First, we compute $\|\chi_{B_R}\|_*$. Note that

$$\begin{aligned} \pi R^2 = |B_R| = \int \chi_{B_R}(x) dx = \int |\chi_{B_R}(x)|^2 dx \leq \|\chi_{B_R}\|_* \int |D\chi_{B_R}| \\ = \|\chi_{B_R}\|_* |\partial B_R| = 2\pi R \|\chi_{B_R}\|_*, \end{aligned}$$

hence

$$\|\chi_{B_R}\|_* \geq \frac{R}{2}.$$

On the other hand, consider the vector-valued function

$$g = \left(\chi_{B_R}(x) - \frac{R^2}{|x|^2} \chi_{B_R^c}(x) \right) \frac{x}{2}.$$

Then, clearly, $\nabla \cdot g = 1$ when $|x| \leq R$, and for $|x| > 1$ we have

$$\nabla \cdot g = -\frac{1}{2} \nabla \cdot \left(\frac{R^2}{|x|^2} x \right) = -\frac{R^2}{2} \nabla \cdot \nabla (\log |x|) = 0.$$

We conclude that $\nabla \cdot g = \chi_{B_R}(x)$. It follows that

$$\|\chi_{B_R}\|_* \leq \|g\|_\infty = \frac{R}{2}.$$

Therefore, we have $\|\chi_{B_R}\|_* = R/2$.

Now, Theorem 2.4 implies that when $\alpha\lambda R/2 \le 1/2$ we have $\bar{u} = 0$ and $\bar{v} = \alpha\chi_{B_R}(x)$. On the other hand, when $\alpha R \ge 1/\lambda$ we may use Theorem 2.5. It says that the minimizer $(\bar{u}, \bar{v})$ is uniquely identified by $\bar{u} + \bar{v} = \chi_{B_R}$, $\|\bar{v}\|_* = 1/(2\lambda)$ and

$$\int \bar{u}(x)\bar{v}(x) = \|\bar{v}\|_* \int |D\bar{u}|.$$

Guessing $\bar{u}(x) = \gamma\chi_{B_R}(x)$ and $\bar{v}(x) = \beta\chi_{B_R}(x)$ we get the following equations for α and β:

$$\gamma + \beta = \alpha, \quad \beta\frac{R}{2} = \frac{1}{2\lambda}, \quad \gamma\beta\pi R^2 = \frac{\beta R}{2}\gamma 2\pi R,$$

which gives $\beta = 1/(\lambda R)$, $\gamma = \alpha - \beta$, as in (2.62), completing the proof.

Let us remark on what happens if the image f is non-negative. Then $\bar{u}$ has to be non-negative as well. Indeed, if $\bar{u}$ is negative somewhere we may consider a trial pair $(|\bar{u}|, f - |\bar{u}|)$ that will satisfy:

$$\int |D|\bar{u}|| \le \int |D\bar{u}|$$

simply because $\bar{u} \in \mathrm{BV}$, and

$$\int (f - |\bar{u}|)^2 dx \le \int (f - \bar{u})^2 dx,$$

simply because $|f - |\bar{u}|| \le |f - \bar{u}|$ since $f \ge 0$. Therefore, $\bar{u} \ge 0$ if $f \ge 0$ everywhere.

A striking consequence of Theorem 2.4 is the following proposition that says that not all images can be obtained as a result of Rudin-Osher denoising.

Proposition 2.7 *Not all functions u in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ can be minimizers of the Rudin-Osher functional for some f.*

Sketch of the proof. This is because of the duality formulation that we have discussed: if u is a minimizer then v is given by

$$v(x) = -\frac{1}{2\lambda}\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right).$$

Take $u(x) = \phi(|x|)$ with a smooth decreasing function $\phi(r)$, then the above means that

$$v(x) = -\frac{1}{2\lambda}\nabla \cdot \left(\frac{x}{|x|}\right) = \frac{1}{2\lambda|x|},$$

which is not in L^2!

The opposite news are given by the next proposition.

Proposition 2.8 *If Ω is a smooth bounded domain then $u(x) = \chi_\Omega(x)$ is a minimizer of the Rudin-Osher functional for some f.*

Proof. According to Theorem 2.5 it suffices to find $v(x)$ such that

$$\int u(x)v(x)dx = \|v\|_* \int |Du|, \tag{2.63}$$

and then set $\lambda = 1/(2\|v\|_*)$, and $f = u + v$. Let us construct a vector-valued function g so that $\|g\|_\infty = 1$ and

$$\int u(x)(\nabla \cdot g(x))dx = \int |Du| = |\partial\Omega|. \tag{2.64}$$

Green's formula means that the latter condition is equivalent to

$$\int_{\partial\Omega} (g \cdot n)dl = |\partial\Omega|. \tag{2.65}$$

This condition holds if g satisfies $g = n$ on $\partial\Omega$. Therefore, we take $g = n$ on $\partial\Omega$, extend it smoothly inside Ω so that $\|g\|_\infty = 1$ and set $v = \nabla \cdot g$ and $f = u + v$. Then we have $\|v\|_* \le 1$. In addition, (2.64) means that

$$\int u(x)v(x)dx = \int |Du|. \tag{2.66}$$

It follows that $\|v\|_* \ge 1$, hence $\|v\|_* = 1$, and (2.63) holds. Therefore, $f = u + v$ is the Rudin-Osher decomposition of f.

Proposition 2.8 can not be generalized too much.

Proposition 2.9 *The product $g(x)\chi_\Omega(x)$ where $g(x)$ is a smooth function and Ω is a smooth domain is not in general the u component in the Rudin-Osher decomposition of any f.*

We refer to Meyer's notes for the proof. Here is a general criterion.

Theorem 2.10 *A pair $(u, v) \in BV \times L^2$ is the Rudin-Osher minimizer for some f if there exists a vector-valued function g such that*

$$|Du| = g \cdot Du, \quad \|g\|_\infty = 1, \quad v = -\nabla \cdot g. \tag{2.67}$$

Sketch of the proof. Let us explain why this is morally equivalent to the conditions in Theorem 2.5, omitting some of the technicalities related to the regularity issues (see Meyer's notes for details). As in the proof of Proposition 2.8 we need to verify that our assumptions on u and v are equivalent to

$$\int u(x)v(x)dx = \|v\|_* \int |\nabla u|, \tag{2.68}$$

and then set $\lambda = 1/(2\|v\|_*)$, and $f = u + v$. Our assumptions on u, v and g imply

$$\int u(x)v(x)dx = -\int u(x)(\nabla \cdot g)dx = \int (Du \cdot g) = \int |Du|. \tag{2.69}$$

It follows that $\|v\|_* \ge 1$ but the fact that $\|g\|_\infty = 1$ means that $\|v\|_* \le 1$, hence $\|v\|_* = 1$, whence (2.69) is equivalent to (2.68), and we are done.

The next corollary of Theorem 2.6 means that the Rudin-Osher functional has trouble reconstructing Lipschitz (as opposed to smooth) domains.

Corollary 2.11 *The indicator function of the unit square $D = [0,1] \times [0,1]$ is not the u component of the minimizer of the Rudin-Osher functional for any f.*

Proof. If (u, v) is a minimizing pair for some f there should exist a vector-valued function g such that $\|g\|_\infty = 1$, $v = -\nabla \cdot g \in L^2$, and $g = n$ a.e. on $\Gamma = \partial D$. Consider the domain

$$D_\varepsilon = \{x \in D : \ 0 \le x_1 + x_2 \le \varepsilon\},$$

and let $\Gamma_\varepsilon = \partial D_\varepsilon$ (a small triangle). Then we have

$$\int_{\Gamma_\varepsilon} (g \cdot n) dl = \int_{D_\varepsilon} (\nabla \cdot g) dx.$$

The integral in the left side is at least $2\varepsilon - \sqrt{2}\varepsilon$, but the integral in the right side is $o(\varepsilon)$ since $v = -\nabla \cdot g$ is an L^2-function. This is a contradiction.

Theorem 2.4 says that if the image f has too small G-norm then the Rudin-Osher funcional wipes it out. Essentially, in that case the Rudin-Osher functional treats the image as pure noise. Here is a useful criterion of when this happens.

Proposition 2.12 *Let f_n be a sequence of functions such that:*
(i) all f_n are supported in a compact set K;
(ii) there exists $q > 2$ and $C > 0$ so that $\|f_n\|_q \le C$;
(iii) $f_n \to 0$ in the sense of distributions.
Then $\|f_n\|_* \to 0$ as $n \to +\infty$.

This happens, for instance, for an image of the form $f_n(x) = p(nx)\chi_\Omega$ where Ω is a bounded domain with a smooth boundary and $p(x)$ is a periodic mean-zero function p. Such images are treated as noise by the Rudin-Osher functional when n is sufficiently large.

3 Image denoising via wavelet shrinkage

In this section we will consider another denoising algorithm, wavelet shrinkage that, in some sense, gives an optimal denoising procedure.

3.1 Wavelets and multiresolution analysis

The Haar wavelets

Let us first recall some very basic facts of the wavelet analysis. The first and simplest example are the Haar wavelets. The basic Haar function is

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2, \\ -1 & \text{if } 1/2 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

It has mean zero

$$\int_0^1 \psi(x) dx = 0,$$

and is normalized so that

$$\int_0^1 \psi^2(x)dx = 1.$$

The rescaled and shifted Haar functions are

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z}.$$

Lemma 3.1 *The Haar functions form an orthonormal set in $L^2(\mathbb{R})$.*

Sketch of the proof. This can be verified by a direct computation – the basic ingredients are that (1) each Haar wavelet has mean zero, (2) the Haar wavelet ψ_{jk_1} and ψ_{jk_2} on the same scale 2^{-j} have disjoint support, and (3) the Haar wavelet ψ_{jk} is constant on the support of $\psi_{j'k}$ if $j < j'$.

The Haar coefficients of a function $f \in L^2(\mathbb{R})$ are defined as the inner products

$$c_{jk} = \int f(x)\psi_{jk}(x)dx, \tag{3.2}$$

and the Haar series of f is

$$\sum_{j,k\in\mathbb{Z}} c_{jk}\psi_{jk}(x). \tag{3.3}$$

Orthonormality of the family $\{\psi_{jk}\}$ ensures that

$$\sum_{j,k} |c_{jk}|^2 \le \|f\|_{L^2}^2 < +\infty,$$

and the series (3.3) converges in $L^2(\mathbb{R})$. In order to show that it actually converges to the function f itself we need to prove that the Haar functions form a basis for $L^2(\mathbb{R})$.

Lemma 3.2 *The Haar wavelets form the basis of $L^2(\mathbb{R})$.*

Sketch of the proof. We consider the dyadic projections P_n defined as follows. Given $f \in L^2(\mathbb{R})$, and $n,k \in \mathbb{Z}$, consider the intervals $I_{nk} = ((k-1)/2^n, k/2^n]$, then

$$P_n f(x) = \fint_{I_{nk}} f dx = 2^n \int_{I_{nk}} f dx, \quad \text{for } x \in I_{nk}.$$

The function $P_n f$ is constant on each of the dyadic intervals I_{nk}. In particular, each Haar function ψ_{jk} satisfies $P_n\psi_{jk}(x) = 0$ for $j \ge n$, while $P_n\psi_{jk}(x) = \psi_{jk}(x)$ for $j < n$. The main observation is that for any $f \in L^2(\mathbb{R})$ we have

$$P_{n+1}f - P_n f = \sum_{k\in\mathbb{Z}} c_{nk}\psi_{nk}(x), \tag{3.4}$$

with the Haar coefficients c_{nk} given by (3.2).

As a consequence of (3.4) we deduce that

$$P_{n+1}f(x) - P_{-m}f(x) = \sum_{j=-m}^{n}\sum_{k\in\mathbb{Z}} c_{jk}\psi_{jk}(x), \tag{3.5}$$

for all $m, n \in \mathbb{Z}$ with $n > m$. The last step is to show that for any $f \in L^2(\mathbb{R})$ we have

$$\lim_{m\to+\infty} P_{-m}f(x) = 0, \qquad \lim_{n\to+\infty} P_n f(x) = f(x), \tag{3.6}$$

both in the L^2-sense. The operators $P_n f$ are uniformly bounded because for all $n, k \in \mathbb{Z}$ we have

$$\int_{I_{nk}} |(P_n f)(x)|^2 dx = 2^{-n}2^{2n}\left|\int_{I_{nk}} f(y)dy\right|^2 \le \int_{I_{nk}} |f(y)|^2 dy.$$

Summing over $k \in \mathbb{Z}$ for a fixed n we get

$$\int_{\mathbb{R}} |P_n f(x)|^2 \le \int_{\mathbb{R}} |f(x)|^2,$$

thus $\|P_n f\|_{L^2} \le \|f\|_{L^2}$. Uniform boundedness of P_n implies that it is sufficient to establish both limits in (3.6) for functions $f \in C_c(\mathbb{R})$, and that is not difficult.

The general MRA framework

A generalization of the above Haar construction is the framework of the multi-resolution analysis (MRA) that is at the heart of the modern theory of wavelets.

Definition 3.3 *An orthonormal MRA of $L^2(\mathbb{R})$ is a chain of closed subspaces*

$$\ldots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \ldots,$$

which satisfies the following properties:

(1) $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$,

(2) *the union* $\bigcup_{j=-\infty}^{\infty} V_j$ *is dense in* $L^2(\mathbb{R})$,

(3) *a function $f(x)$ belongs to V_j if and only if $f(2x)$ belongs to V_{j+1},*

(4) *there is a function $\phi \in V_0$ so that $\{\phi(x-k), k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.*

The function $\phi(x)$ is called the scaling function.

The basic intuition behind this construction is that the space V_j contains functions that oscillate, roughly, on the scale 2^{-j}. In the case of the Haar wavelets, this space was made of functions that are piecewise constant on intervals of the form $[k/2^j, (k+1)/2^j)$.

We will now establish some basic facts that follow from the definition of MRA. An immediate consequence is that the functions

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), \quad k \in \mathbb{Z},$$

form an orthonormal basis of the space V_j. In particular, $\phi(x)$ has a representation (since $\phi \in V_0 \subset V_1$, and $\phi(2x-k)$ form a basis for V_1):

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} h_k \phi(2x-k) = \sum_{k\in\mathbb{Z}} (\sqrt{2}h_k)\phi_{1,k}, \tag{3.7}$$

with the coefficients h_k given by

$$h_k = \frac{1}{\sqrt{2}}\langle \phi, \phi_{1,k} \rangle, \quad k \in \mathbb{Z}. \tag{3.8}$$

As we will soon see, these coefficients have to satisfy a number of interesting relations.

Another simple observation is that if $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$\int_{\mathbb{R}} \phi(x)dx \neq 0, \tag{3.9}$$

which means, in a sense, that the scaling ϕ is not oscillatory. To see that, assume for simplicity that ϕ is compactly supported. Then, if $\phi(x)$ has integral zero, we have

$$\int_{\mathbb{R}} \phi_{j,k}(x)dx = 0,$$

for all $j, k \in \mathbb{Z}$. Consider any function $u \in L^2(\mathbb{R})$ that is compactly supported and with

$$\int_{\mathbb{R}} u(x)dx = 1.$$

Let P_j be the orthogonal projection on the space V_j, and $u_j = Pu_j$. Let us define the coefficients

$$g_{jk} = \int u(x)\phi_{j,k}(x)dx.$$

As all $\phi_{j,k}$ lie in the space V_j, we have $P_j\phi_{j,k} = \phi_{j,k}$, and since the operator P_j is self-adjoint, we have

$$g_{jk} = \int u(x)\phi_{j,k}(x)dx = \int u(x)P_j\phi_{j,k}dx = \int (P_j u)\phi_{j,k}dx = \int u_j\phi_{j,k}dx.$$

Since both u and ϕ are compactly supported, it follows that for any j fixed, only finitely many of the coefficients g_{jk} are non-zero. Hence, u_j satisfies

$$\int_{\mathbb{R}} u_j(x)dx = \sum_j \int g_{jk}\phi_{jk}(x)dx = 0,$$

as the above sum has only finitely many terms, and interchanging summation and integration is not an issue. We also know that u_j converge strongly to u in $L^2(\mathbb{R})$ as $j \to +\infty$. As all u_j are uniformly compactly supported for $j \geq 1$, we deduce that u_j converges strongly to u in $L^1(\mathbb{R})$ which implies automatically that

$$\int_{\mathbb{R}} u(x)dx = \lim_{j\to+\infty} \int_{\mathbb{R}} u_j(x)dx = 0,$$

which is a contradiction. Hence, (3.9) holds.

Next, identity (3.7) implies that the Fourier transform of ϕ satisfies

$$\hat{\phi}(\omega) = H\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right), \tag{3.10}$$

with a periodic function

$$H(\omega) = \sum_{k\in\mathbb{Z}} e^{ik\cdot\omega} h_k, \tag{3.11}$$

whose Fourier coefficients h_k are given by (3.8) (and are, therefore, determined by the scaling function ϕ). Iterating relation (3.10) gives

$$\hat{\phi}(\omega) = \hat{\phi}(0) \prod_{j=1}^{\infty} H\left(\frac{\omega}{2^j}\right), \tag{3.12}$$

so that the scaling function is completely determined by the coefficients h_k. If, in addition to the L^2-normalization of ϕ we have

$$\int_{\mathbb{R}} \phi(x)dx = 1,$$

then $\hat{\phi}(0) = 1$ and (3.12) becomes

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} H\left(\frac{\omega}{2^j}\right). \tag{3.13}$$

Let us now formulate the orthonormality condition on the basis in terms of the function $H(\omega)$. Consider the 2π-periodic function

$$A(\omega) = \sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega + 2n\pi)|^2,$$

then the two-scale relation (3.10) implies that

$$A(2\omega) = \sum_{n} |\hat{\phi}(2\omega + 2n\pi)|^2 = \sum_{n} |H(\omega + n\pi)|^2 |\hat{\phi}(\omega + n\pi)|^2.$$

Using the fact that $H(\omega)$ is 2π-periodic we may re-write this as

$$A(2\omega) = \sum_{m\in\mathbb{Z}} |H(\omega)|^2 |\hat{\phi}(\omega + 2m\pi)|^2 + \sum_{m\in\mathbb{Z}} |H(\omega + \pi)|^2 |\hat{\phi}(\omega + (2m+1)\pi)|^2.$$

Therefore, the function $A(\omega)$ satisfies an identity

$$A(2\omega) = |H(\omega)|^2 A(\omega) + |H(\omega + \pi)|^2 A(\omega + \pi). \tag{3.14}$$

On the other hand, orthonormality of $\phi(x-k)$ implies that for any $k \in \mathbb{Z}$ we have ($*$ denotes the complex conjugation)

$$\begin{aligned} \delta_{0,k} &= \int_{\mathbb{R}} \phi(x)\phi^*(x+k)dx = \frac{1}{(2\pi)^2} \int e^{i\omega x - i\omega'(x+k)} \hat{\phi}(\omega)\hat{\phi}^*(\omega')dxd\omega d\omega' \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\phi}(\omega)|^2 e^{-i\omega k} d\omega. \end{aligned} \tag{3.15}$$

The integral in the right side is nothing but

$$\int_0^{2\pi} A(\omega) e^{-i\omega k} d\omega,$$

hence (3.15) simply says that the Fourier coefficients of $A(\omega)$ are $\delta_{0,k}$. This means that

$$A(\omega) = \sum_{n\in\mathbb{N}} |\hat{\phi}(\omega + 2\pi n)|^2 \equiv 1, \tag{3.16}$$

and (3.14) implies the following condition on $H(\omega)$:

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1. \tag{3.17}$$

We emphasize that relation (3.17) was obtained simply from the definition of an MRA, and is, therefore, an algebraic restriction on the scaling function $\phi(x)$. It is useful to summarize the above computation as follows.

Proposition 3.4 *Let $\phi(x) \in L^2(\mathbb{R})$, then the functions $\phi(x-k)$, $k \in \mathbb{Z}$ form an orthonormal set if and only if*

$$A(\omega) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 \equiv 1. \tag{3.18}$$

So far we have only dealt with the scaling function. Let us now turn to wavelets. Consider the spaces V_j and V_{j+1}, and write

$$V_{j+1} = V_j \oplus W_j,$$

where W_j is the orthogonal complement of V_j in V_{j+1}. The usual interpretation is that W_j is the space of details that are lost when passing from the "fine" space V_{j+1} to the "coarse" space V_j. It is easy to see that W_j inherits the scaling property of V_j: if a function $u(x)$ is in W_j then the function $u(2x)$ is in W_{j+1}.

As a consequence of completeness of V_j, any function $u(x)$ can be decomposed as

$$u(x) = \sum_{j\in\mathbb{Z}} u_j(x),$$

with each $u_j \in W_j$. One can think of u_j as the details of u on the scale 2^{-j}.

Let us now pass from the scaling function $\phi(x)$ (the "father wavelet") to the "mother wavelet" $\psi(x)$. This is done as follows. Given a function $\eta \in W_0$, as it lies in $V_1 = V_0 \oplus W_0$, we can decompose $\eta(x)$ as

$$\eta(x) = 2\sum_k g_k \phi(2x-k), \quad g_k = \frac{1}{2}\int \eta(x)\phi(2x-k)dx, \tag{3.19}$$

and in the Fourier domain:

$$\begin{aligned}\hat{\eta}(\omega) &= 2\sum_{k\in\mathbb{Z}} \int e^{-i\omega x} g_k \hat{\phi}(\omega') e^{i\omega'(2x-k)} \frac{dx d\omega'}{2\pi} = \sum_{k\in\mathbb{Z}} g_k \int e^{-i\omega x/2} \hat{\phi}(\omega') e^{i\omega'(x-k)} \frac{dx d\omega'}{2\pi} \\ &= \hat{\phi}\left(\frac{\omega}{2}\right) \sum_k g_k e^{-i\omega k/2} = \hat{\phi}\left(\frac{\omega}{2}\right) G\left(\frac{\omega}{2}\right),\end{aligned} \tag{3.20}$$

with a 2π-periodic function

$$G(\omega) = \sum_{k\in\mathbb{Z}} g_k e^{-ik\omega}. \tag{3.21}$$

Moreover, as $\eta \in W_0$, and $\phi(x-k) \in V_0$ for all $k \in \mathbb{Z}$, we have the orthogonality condition

$$\int_{\mathbb{R}} \eta(x)\phi^*(x-k)dx = 0, \quad \text{for all } k \in \mathbb{Z},$$

or, on the Fourier side:

$$\int_{\mathbb{R}} \hat{\eta}(\omega)\hat{\phi}^*(\omega)e^{-ik\omega}d\omega = 0 \quad \text{for all } k \in \mathbb{Z}.$$

This can be re-written as

$$\sum_{n\in\mathbb{Z}} \int_0^{2\pi} \hat{\eta}(\omega+2\pi n)\hat{\phi}^*(\omega+2\pi n)e^{-ik\omega}d\omega = 0 \quad \text{for all } k \in \mathbb{Z},$$

meaning that

$$\sum_{n\in\mathbb{Z}} \hat{\eta}(\omega+2\pi n)\hat{\phi}^*(\omega+2\pi n) = 0 \text{ for a.e. } \omega \in [0,2\pi]. \tag{3.22}$$

Using relation (3.20) in (3.22), and replacing ω by 2ω gives

$$0 = \sum_{n\in\mathbb{Z}} G(\omega+\pi n)\hat{\phi}(\omega+\pi n)\hat{\phi}^*(2\omega+2\pi n). \tag{3.23}$$

Now, we can use relation (3.10) to replace the last $\hat{\phi}^*$ above:

$$0 = \sum_{n\in\mathbb{Z}} G(\omega+\pi n)H^*(\omega+\pi n)|\hat{\phi}(\omega+\pi n)|^2, \tag{3.24}$$

with $H(\omega)$ defined by (3.11). Splitting the above sum into even and odd n and using 2π-periodicity of the functions $G(\omega)$ and $H(\omega)$, we obtain:

$$0 = G(\omega)H^*(\omega)\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega+2\pi n)|^2 + G(\omega+\pi)H^*(\omega+\pi)\sum_{n\in\mathbb{Z}} |\hat{\phi}(\omega+\pi+2n\pi)|^2. \tag{3.25}$$

Now, identity (3.16) implies that

$$G(\omega)H^*(\omega) + G(\omega+\pi)H^*(\omega+\pi) = 0. \tag{3.26}$$

Let us recapitulate: starting with any function $\eta \in W_0$ we have defined the coefficients g_k via the decomposition (3.19) and formed the function $G(\omega)$ (3.21) whose Fourier coefficients are g_k. Identity (3.26) holds for any function $G(\omega)$ obtained in this way.

Define now

$$\lambda(\omega) = e^{i\omega}G(\omega)/H^*(\omega+\pi),$$

then (3.26) becomes $\lambda(\omega) = \lambda(\omega+\pi)$. That is, the function $\lambda(\omega)$ is π-periodic, and the function $C(\omega) = \lambda(\omega/2)$ is 2π-periodic. If we also set

$$G_0(\omega) = e^{-i\omega}H^*(\omega+\pi),$$

then

$$G(\omega) = e^{-i\omega}\lambda(\omega)H^*(\omega+\pi) = G_0(\omega)C(2\omega),$$

and, finally, we get, going back to (3.20):

$$\hat{\eta}(\omega) = \hat{\phi}\left(\frac{\omega}{2}\right)G\left(\frac{\omega}{2}\right) = C(\omega)G_0\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) = C(\omega)\hat{\psi}\left(\omega\right), \tag{3.27}$$

with

$$\hat{\psi}(\omega) = G_0\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right). \tag{3.28}$$

We have finally arrived to our destination: the function

$$\psi(x) = \int \hat{\psi}(\omega)e^{i\omega x}\frac{d\omega}{2\pi} = \int G_0\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)e^{i\omega x}\frac{d\omega}{2\pi} = \int e^{-i\omega/2}H^*\left(\frac{\omega}{2}+\pi\right)\hat{\phi}\left(\frac{\omega}{2}\right)e^{i\omega x}\frac{d\omega}{2\pi} \tag{3.29}$$

is called the "mother wavelet". We have also discovered that any function $\eta \in W_0$ has a representation

$$\begin{aligned}\eta(x) &= \int_{\mathbb{R}} e^{i\omega x}\hat{\eta}(\omega)\frac{d\omega}{2\pi} = \int_{\mathbb{R}} e^{i\omega x}C(\omega)\hat{\psi}(\omega)\frac{d\omega}{2\pi} = \sum_{k\in\mathbb{Z}} c_k \int_{\mathbb{R}} e^{i\omega x - ik\omega}\hat{\psi}(\omega)\frac{d\omega}{2\pi} \\ &= \sum_{k\in\mathbb{N}} c_k\psi(x-k).\end{aligned} \tag{3.30}$$

Here c_k are the Fourier coefficients of the periodic function $C(\omega)$:

$$C(\omega) = \sum_{k\in\mathbb{N}} c_k e^{-ik\omega}.$$

Representation (3.30) is an important result: it shows that the translations $\psi(x-k)$ of the mother wavelet span the space W_0.

Let us now verify that $\psi(x-k)$ form an orthonormal basis for W_0 – we only need to check that they form an orthonormal set, as we have already shown that they span W_0. If we set

$$B(\omega) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\omega+2\pi n)|^2, \tag{3.31}$$

then according to Proposition 3.4 it suffices to verify that

$$B(\omega) \equiv 1. \tag{3.32}$$

As in our computation for the scaling function when we showed that $A(\omega) \equiv 1$, (3.32) is equivalent to the orthonormality of $\psi(x-n)$. Note that

$$\begin{aligned}B(2\omega) &= \sum_{k\in\mathbb{Z}} |\hat{\psi}(2\omega+2\pi n)|^2 = \sum_{k\in\mathbb{Z}} |G_0(\omega+\pi n)|^2|\hat{\phi}(\omega+\pi n)|^2 \\ &= |G_0(\omega)|^2A(\omega) + |G_0(\omega+\pi)|^2A(\omega+\pi) = |G_0(\omega)|^2 + |G_0(\omega+\pi)|^2 \\ &= |H(\omega+\pi)|^2 + |H(\omega)|^2 = 1,\end{aligned}$$

because of (3.17). We conclude that $\psi(x-k)$ form an orthonormal basis for W_0. Using the scaling we deduce that $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, with j fixed, form an orthonormal basis for W_j. It follows, finally, that $\psi_{j,k}$, with $j, k \in \mathbb{Z}$ form an orthonormal basis for $L^2(\mathbb{R})$.

Next, we show that the mother wavelet has mean-zero (and hence, has to be oscillatory, unlike the scaling function $\phi(x)$)

$$\int_{\mathbb{R}} \psi(x)dx = 0. \tag{3.33}$$

This is seen as follows. As the mother wavelet lives in the space V_0, it can be decomposed as

$$\psi(x) = 2\sum_k g_k \phi(2x-k).$$

Then the Fourier transform

$$G(\omega) = \sum_{k\in\mathbb{Z}} g_k e^{ik\omega}$$

has to satisfy (3.26), as we have shown above, for any function in W_0:

$$G(\omega)H^*(\omega) + G(\omega+\pi)H^*(\omega+\pi) = 0. \tag{3.34}$$

If the scaling function is normalized so that

$$\int_{\mathbb{R}} \phi(x)dx = 1,$$

then $H(0) = 1$, which implies $H(\pi) = 0$ because of (3.17):

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1.$$

This, in turn, implies that $G(0) = 0$, and thus (3.33) holds.

An alternative point of view is to start with a periodic function $H(\omega)$ that satisfies (3.17)

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1, \tag{3.35}$$

and a periodic function $G(\omega)$ that satisfies

$$G(\omega)H^*(\omega) + G(\omega+\pi)H^*(\omega+\pi) = 0. \tag{3.36}$$

Given the function $H(\omega)$, we construct the scaling function $\phi(x)$ via (3.13):

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} H\left(\frac{\omega}{2^j}\right), \tag{3.37}$$

and the mother wavelet via

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} g_k \phi(2x-k), \tag{3.38}$$

where

$$G(\omega) = \sum_{k\in\mathbb{Z}} g_k e^{ik\omega}.$$

The Meyer wavelets

Consider the scaling function ϕ defined by its Fourier transform

$$\hat{\phi}(\omega) = \begin{cases} 1, & |\omega| \leq 2\pi/3, \\ \cos\left[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\omega| - 1)\right], & 2\pi/3 \leq |\omega| \leq 4\pi/3, \\ 0, & \text{otherwise.} \end{cases} \tag{3.39}$$

The function $\nu(\omega)$ satisfies

$$\nu(s) = \begin{cases} 0, & \text{if } s \leq 0, \\ 1, & \text{if } s \geq 1, \end{cases} \tag{3.40}$$

with the additional property

$$\nu(s) + \nu(1 - s) = 1. \tag{3.41}$$

The function $\hat{\phi}(\omega)$ is continuous and has the form of a bump centered at $\omega = 0$ with $\hat{\phi}(0) = 1$ and $0 \leq \hat{\phi}(\omega) \leq 1$. The property (3.41) of the function ν implies that

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi k)|^2 = 1. \tag{3.42}$$

To see this, notice first that the sum in (3.42) is 2π-periodic in ω so that we may assume without loss of generality that $\omega \in [-\pi, \pi]$ and, second, that since the support of $\hat{\phi}$ is the interval $[-4\pi/3, 4\pi/3]$, given any $\omega \in \mathbb{R}$, only two terms in this sum do not vanish. If $\omega \in [-2\pi/3, 2\pi/3]$ then actually only one term is non-zero: $\hat{\phi}(\omega) = 1$ and $\hat{\phi}(\omega + 2\pi k) = 0$ for all $k \neq 0$. On the other hand, say for $2\pi/3 \leq \omega \leq \pi$ we have $-4\pi/3 \leq \omega - 2\pi \leq -\pi < -2\pi/3$, and

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} |\hat{\phi}(\omega + 2\pi k)|^2 = |\hat{\phi}(\omega)|^2 + |\hat{\phi}(\omega - 2\pi)|^2 \\ &= \cos^2(\frac{\pi}{2}\nu(\frac{3}{2\pi}|\omega| - 1)) + \cos^2(\frac{\pi}{2}\nu(\frac{3}{2\pi}|\omega - 2\pi| - 1)) \\ &= \cos^2(\frac{\pi}{2}\nu(\frac{3}{2\pi}\omega - 1)) + \cos^2(\frac{\pi}{2}\nu(\frac{3}{2\pi}(2\pi - \omega) - 1)) \\ &= \cos^2(\frac{\pi}{2}\nu(\frac{3}{2\pi}\omega - 1)) + \cos^2(\frac{\pi}{2}\nu(2 - \frac{3}{2\pi}\omega) = 1, \end{aligned}$$

by (3.41) with $s = (\frac{3}{2\pi}\omega - 1)$. Identity (3.42) in turn implies that the functions $\phi(x - k)$, $k \in \mathbb{Z}$, form an orthonormal set. We then define V_0 as the closed subspace spanned by this set, and V_j as the closed subspace spanned by $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$. In order to verify that $V_j \subseteq V_{j+1}$ it suffices to check that $\phi \in V_1$. As we have seen in our general analysis of MRA, this is equivalent to the existence a 2π-periodic function $H(\omega)$ so that

$$\hat{\phi}(\omega) = H(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}). \tag{3.43}$$

Let us verify that this relation holds with $H(\omega)$ defined by

$$H(\omega) = \sum_{k\in\mathbb{Z}} \hat{\phi}(2\omega + 4\pi k). \tag{3.44}$$

This function is 2π-periodic and

$$H(\omega/2)\hat{\phi}(\omega/2) = \sum_{k\in\mathbb{Z}} \hat{\phi}(\omega + 4\pi k)\hat{\phi}(\omega/2) = \hat{\phi}(\omega)\hat{\phi}(\omega/2) \tag{3.45}$$

because $\hat{\phi}(\omega)$ is supported on $[-4\pi/3, 4\pi/3]$. Moreover, if $\hat{\phi}(\omega) = 0$ but $\hat{\phi}(\omega/2) \neq 0$, that is, when $4\pi/3 \leq |\omega| < 8\pi/3$, then $\hat{\phi}(\omega + 4\pi k) = 0$ for all $k \in \mathbb{Z}$ so that $H(\omega/2) = 0$. On the other hand, if ω is in the support of $\hat{\phi}$ then $\hat{\phi}(\omega/2) = 1$, and (3.45) implies that

$$H(\omega/2)\hat{\phi}(\omega/2) = \hat{\phi}(\omega). \tag{3.46}$$

It is easy to check that the spaces V_j defined in this way satisfy the other requirements of an MRA. The mother wavelet is given by

$$\hat{\psi}(\omega) = e^{-i\omega/2}H^*(\frac{\omega}{2} + \pi)\hat{\phi}(\frac{\omega}{2}) = e^{-i\omega/2}[\hat{\phi}(\omega + 2\pi) + \hat{\phi}(\omega - 2\pi)]\hat{\phi}(\omega/2). \tag{3.47}$$

Tensor product wavelets

The simplest way to extend the MRA and wavelets to 2D is to treat $L^2(\mathbb{R}^2)$ as a tensor product $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

Theorem 3.5 *We have $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, that is, for any $f(x_1, x_2) \in L^2(\mathbb{R}^2)$ and any $\varepsilon > 0$ there exists a function $g_\varepsilon(x_1, x_2)$ of the form*

$$g_\varepsilon(x_1, x_2) = \sum_{k=1}^{N} c_k g_k(x_1) e_k(x_2),$$

with $g_k, e_k \in L^2(\mathbb{R})$ and $c_k \in \mathbb{R}$ so that

$$\|f - g_\varepsilon\|_{L^2(\mathbb{R}^2)} < \varepsilon.$$

Proof. It suffices to verify the claim of the theorem for smooth test functions $\phi \in C_c^\infty(\mathbb{R}^2)$. Without loss of generality we may assume that the function $\phi(x_1, x_2)$ is supported in the open square $Q = (-\pi, \pi) \times (-\pi, \pi)$. As ϕ vanishes near the boundary of Q, we may extend ϕ periodically to a function $\Phi(x_1, x_2)$ that is 1-periodic in both variables. The function $\Phi(x_1, x_2)$ can be approximated in $L^2(Q)$ by finite sums of its Fourier series

$$\Phi_N(x_1, x_2) = \sum_{k,m=-N}^{N} c_{nm} e^{inx_1} e^{imx_2},$$

that is, for N sufficiently large we have

$$\|\Phi - \Phi_N\|_{L^2(Q)} < \varepsilon.$$

Then setting $g_k(x_1) = \chi_{[-\pi,\pi]}(x_1)e^{ikx_1}$ and $e_k(x_2) = \chi_{[-\pi,\pi]}(x_2)e^{ikx_2}$ we obtain that

$$\|\phi - \sum_{k,m=-N}^{N} c_{nm} g_n(x_1) e_m(x_2)\|_{L^2(\mathbb{R}^2)} < \varepsilon,$$

which is the estimate we need.

Now, in order to construct an MRA of $L^2(\mathbb{R}^2)$ we start with an MRA of $L^2(\mathbb{R})$:

$$\ldots \subseteq V_{j-1} \subseteq V_j \subseteq V_{j+1} \subseteq \ldots$$

Consider the tensor products

$$V_{(j)} = V_j \otimes V_j = \text{closure}\{\text{span}[f_1(x_1)f_2(x_2)] : \ f_1, f_2 \in V_j\}.$$

Since $V_{j-1} \subseteq V_j$, we have $V_{(j-1)} \subseteq V_{(j)}$. The scaling relation transfers to $V_{(j)}$ also: if $f(x)$ is in $V_{(j)}$, $f(2x)$ is in $V_{(j+1)}$. Then we define the two-dimensional scaling function as

$$\psi^{(0,0)}(x_1, x_2) = \phi(x_1)\phi(x_2), \tag{3.48}$$

and the three mother wavelets by

$$\begin{aligned} &\psi^{(0,1)}(x_1, x_2) = \phi(x_1)\psi(x_2), \quad \psi^{(1,0)}(x_1, x_2) = \psi(x_1)\phi(x_2), \\ &\psi^{(1,1)}(x_1, x_2) = \psi(x_1)\psi(x_2). \end{aligned} \tag{3.49}$$

In order to go to other scales, given any scale index $j \in \mathbb{Z}$ and any integer vector $k = (k_1, k_2)$, with $k_{1,2} \in \mathbb{Z}$, we set

$$\psi^{\alpha}_{j,k}(x) = 2^j \psi^{\alpha}(2^j x - k), \tag{3.50}$$

where α is one of the multi-indices $(0,0), (0,1), (1,0), (1,1)$. Then the 1D MRA properties imply immediately that at any fixed scale j the functions $\psi^{(0,0)}_{j,k}$, $k \in \mathbb{Z}^2$ form an orthonormal basis for $V_{(j)}$.

Furthermore, as in 1D we may define $W_{(j)}$ as the orthogonal complement of $V_{(j)}$ in $V_{(j+1)}$. The basis of this space is

$$\{\psi^{(0,1)}_{j,k}, \ \psi^{(1,0)}_{j,k}, \ \psi^{(1,1)}_{j,k} : \ k \in \mathbb{Z}^2\}. \tag{3.51}$$

Wavelet decomposition of smooth functions

Theorem 3.6 *Suppose that the mother wavelet $\psi(x)$ is in $C^r(\mathbb{R})$, with some integer $r > 0$, and, in addition,*

$$\int_{\mathbb{R}} (1 + |x|^r)|\psi(x)|dx < +\infty,$$

and $|\psi(x)| \leq C(1 + |x|)^r$ with some $C > 0$. Then all moments of ψ up to order r vanish:

$$\int_{\mathbb{R}} x^k \psi(x)dx = 0, \quad k = 0, 1, \ldots, r. \tag{3.52}$$

The proof of this theorem can be found in Daubechies book "Ten Lectures on Wavelets" (Theorem 5.5.1). Here we just explain the idea why this is true. Let us assume for convenience that $\psi(x)$ is compactly supported. Choose j very large and positive and j' very negative so that ψ_{jk} look very concentrated and $\psi_{j',k'}$ look very spread out. The support of ψ_{jk} is tiny and $\psi_{j'k'}$ is smooth there. Therefore, we may replace $\psi_{j'k'}$ by its Taylor series expanded to as high order as allowed by the smoothness of ψ. The orthogonality of ψ_{jk} and $\psi_{j'k}$ shows that

$$\int \psi_{jk}(x)P_m(x)dx = 0, \tag{3.53}$$

for some polynomial of order m, which in turn implies that

$$\int \psi(x)Q_m(x)dx = 0, \tag{3.54}$$

with some polynomial $Q_m(x)$ of the same order. Varying the location, that is, changing k and also j we get a sufficiently large collection of polynomials $Q_m(x)$ for which (3.54) holds, to conclude that (3.52) holds.

Corollary 3.7 *A mother wavelet that decays at an exponential rate $|\psi(x)| \leq Ce^{-\alpha|x|}$ with some $\alpha > 0$, can not be C^∞.*

Proof. If $\psi(x)$ has an exponential decay then the Fourier transform

$$\hat{\psi}(\omega) = \int \psi(x)e^{-i\omega x}dx$$

is analytic in a strip $D = \{|\mathrm{Im}\omega| \leq \alpha/2\}$. On the other hand, if ψ is a C^∞ mother wavelet then all moments vanish:

$$\int_{\mathbb{R}} x^k\psi(x)dx = 0, \quad k = 0, 1, 2, \ldots$$

This, in turn, implies that

$$\frac{d^k}{d\omega^k}\hat{\psi}|_{\omega=0} = 0.$$

Together with the analyticity of $\hat{\psi}(\omega)$ in the strip D that includes the point $\omega = 0$, this implies that $\psi \equiv 0$.

Corollary 3.8 *A compactly supported mother wavelet can not be C^∞.*

Let us now investigate the wavelet coefficients of a smooth function. Let us assume that the mother wavelet $\psi(x)$ is C^r regular, and suppose that $u(x)$ is a C^m-function with $0 \leq m \leq r+1$. Intuitively, if $u(x)$ is smooth, it is nearly constant on the support of $\psi_{jk}(x)$ for j large, which makes the wavelet coefficients

$$c_{jk} = \int_{\mathbb{R}} u(x)\psi_{jk}(x)dx$$

small simply because

$$\int_{\mathbb{R}} \psi_{jk}(x)dx = 0.$$

In order to make this statement quantitative, let us write c_{jk} as

$$c_{jk} = \int_{\mathbb{R}} u(x)\psi_{jk}(x)dx = 2^{-j/2}\int \psi(y)u(x_0 + 2^{-j}y)dy, \tag{3.55}$$

with $x_0 = k/2^j$. Consider Taylor's expansion around x_0:

$$u(x_0 + t) = \sum_{n=0}^{m-1} \frac{u^{(n)}(x_0)}{n!}t^n + \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}u^{(m)}(x_0 + s)ds = P_{m-1}(t) + R_{m-1}(t).$$

The residual $R_{m-1}(t)$ can be bounded as

$$|R_{m-1}(t)| \leq \frac{\|u\|_{C^m}}{m!}|t|^m.$$

Moreover, as $P_{m-1}(t)$ is a polynomial in t, the corresponding integral in the definition (3.55) of c_{jk} is a sum of moments of ψ of order at most $m-1 \leq r$, hence the integral vanishes, so that

$$c_{jk} = 2^{-j/2} \int \psi(y) R_{m-1}(2^{-j}y) dy, \tag{3.56}$$

and thus the Fourier coefficients of a C^m smooth function satisfy

$$|c_{jk}| \leq 2^{-j/2} \int \psi(y) |R_{m-1}(2^{-j}y)| dy \leq \frac{2^{-(m+1/2)j}\|u\|_{C^m}}{m!} M_m, \tag{3.57}$$

where

$$M_m = \int_{\mathbb{R}} |y|^m \psi(y) dy$$

is the m-th absolute moment of ψ.

Wavelet coefficients of functions with jump discontinuities

In order to understand how the wavelet coefficients "react" to a jump, consider the Heaviside function $u(x) = H(x - x_e)$, so that $u(x) = 1$ for $x \geq x_e$, and $u(x) = 0$ for $x < x_e$. We also assume that the mother wavelet is compactly supported in an interval $[a, b]$, and denote $x_{jk} = k/2^j$ and $h_j = 1/2^j$. Then the wavelet coefficients are

$$\begin{aligned} c_{jk} &= \int_{\mathbb{R}} \psi_{jk}(x) u(x) dx = 2^{j/2} \int \psi(2^j x - k) u(x) dx = 2^{-j/2} \int \psi(x-k) u(\frac{x}{2^j}) dx \\ &= 2^{-j/2} \int_a^b \psi(x) u(\frac{k}{2^j} + \frac{x}{2^j}) dx = \sqrt{h_j} \int_a^b \psi(y) u(x_{jk} + h_j y) dy \\ &= \sqrt{h_j} \int_a^b \psi(y) H(x_{jk} - x_e + h_j y) dy. \end{aligned}$$

Therefore, if

$$x_{jk} - x_e + h_j y \neq 0 \text{ for all } y \in [a, b],$$

then $c_{jk} = 0$ since then $H(x_{jk} - x_e + h_j y)$ is constant on $[a, b]$, and the integral of ψ vanishes. Otherwise we have the estimate

$$|c_{jk}| \leq \sqrt{h_j} \|\psi\|_{L^1}. \tag{3.58}$$

Let us see for how many k (with j fixed) we have

$$x_{jk} - x_e + h_j y = 0 \text{ for some } y \in [a, b].$$

This condition is equivalent to

$$a \leq \frac{x_e - x_{jk}}{h_j} \leq b,$$

or

$$a - 2^j x_e \le -k \le b - 2^j x_e.$$

Therefore, the number of such k is at most $N = [b-a] + 1$, which does not depend on j.

If a function $u(x)$ has a jump at a point x_e and is smooth away from it, then at each level j it generates at most N coefficients that decay as in (3.58) and the rest decay as in the estimate (3.57) for the wavelet coefficients of a smooth function.

Besov spaces and wavelets

As we have seen in the Osher-Rudin functional theory, the space of BV functions is natural in imaging. However, it does not fit perfectly in the wavelet theory, as it can be shown that the space BV can not be characterized by the size of the wavelet coefficients (see Meyer's notes).

On the other hand, wavelet decompositions are particularly suitable for Besov spaces which are defined as follows. Take $u(x) \in L^p(\mathbb{R})$ and for any $h > 0$ define the p-modulus of continuity of u as

$$\omega_p(u,h) = \sup_{|a| \le h} \|u(x+a) - u(x)\|_{L^p(\mathbb{R})}. \tag{3.59}$$

A function u is in the Besov space $B^\alpha_{p,q}$ (with $0 < \alpha \le 1$) if

$$|||u||| = \left(\int_0^\infty \frac{\omega_p(u,h)^q}{h^{\alpha q}} \frac{dh}{h} \right)^{1/q} < +\infty. \tag{3.60}$$

When $q = \infty$, the integral over h above is replaced by the essential supremum over h, as usual. The norm in this space is given by

$$\|u\|_{B^\alpha_q(L^p)} = \|u\|_{L^p} + |||u|||, \tag{3.61}$$

and the homogeneous Besov space $\dot{B}^\alpha_{p,q}$ consists of all functions with a finite $|||u|||$.

When $\alpha > 1$ then the definition of the Besov space $B^\alpha_{p,q}$ is similar except we decompose $\alpha = n + s$ with $n \in \mathbb{N}$ and $s \in (0,1]$ and set

$$|||u||| = \left(\int_0^\infty \frac{\omega_p(u^{(n)},h)^q}{h^{sq}} \frac{dh}{h} \right)^{1/q} < +\infty, \tag{3.62}$$

and

$$\|u\|_{B^\alpha_q(L^p)} = \|u\|_{W^{n,p}} + |||u|||. \tag{3.63}$$

Here $W^{n,,p}$ is the Sobolev space of functions with n derivatievs in L^p.

Let us see what happens in the integral in the definition of $|||u|||$ as $h \downarrow 0$. Let us assume that $\omega_p(u,h) \sim O(h^\beta)$, then for the integral to converge we need

$$\beta q - \alpha q - 1 > -1,$$

that is, $\beta > \alpha$. For this reason α is called the regularity index. The space $B^\alpha_{\infty,\infty}$ consists of functions for which

$$\sup_{h>0} \frac{\omega_\infty(u,h)}{h^\alpha} < +\infty,$$

in other words:

$$\sup_{x,y\in\mathbb{R}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < +\infty.$$

This means that $B^\alpha_{\infty,\infty}$ is simply the Hölder space $C^{0,\alpha}$.

The basic idea behind the above definition is that Besov spaces capture the local irregularities of the function $u(x)$ via the continuity modulus $\omega_p(u,h)$, with the regularity index α capturing the nature of an irregularity, and the indices p and q giving various ways of measurement. The wavelet coefficients are also good at capturing the local behavior at particular scales, and we will now connect these two measures of local oscillations.

We may also define the continuity modulus $\omega_p(u,j)$ only for the discrete scales:

$$\omega_p(u,j) = \sup_{|a|\le 2^{-j}} \|u(x+a)-u(x)\|_{L^p(\mathbb{R})}. \tag{3.64}$$

Since the function $\omega_p(u,h)$ is monotonically increasing in h, the Besov condition (3.60) may be written as (note that when we set $h=2^{-j}$ and discretize the integral as summation over j, the term dh/h goes into the Jacobian: $dh \sim 2^{-j}$ so $dh/h \sim 1$):

$$\sum_{j\in\mathbb{Z}} \omega_p(u,j)^q 2^{\alpha q j} < +\infty. \tag{3.65}$$

We also have $\omega_p(u,j) \le 2\|u\|_{L^p}$ so that

$$\sum_{j<0} \omega_p(u,j)^q 2^{\alpha q j} \le \frac{2^q}{2^{\alpha q}-1}\|u\|^q_{L^p}. \tag{3.66}$$

Therefore, the Besov norm is equivalent to

$$\|u\|_{L^p} + \left(\sum_{j\ge 0} 2^{\alpha q j}\omega_p(u,j)^q\right)^{1/q}. \tag{3.67}$$

This dyadic definition of the Besov spaces leads to a natural connection to wavelets. Consider an MRA

$$\ldots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \ldots,$$

and let E_j and D_j denote the orthogonal projections on the spaces V_j and W_j, respectively. Given a function $u(x)$ let $u_j = D_j u$, and decompose it as

$$u_j(x) = \sum_{k\in\mathbb{Z}} d_{jk}\psi_{jk}(x). \tag{3.68}$$

One may show that the Besov norm is actually equivalent to

$$\|E_0 u\|_{L^p} + \left(\sum_{j\ge 0} 2^{\alpha q j}\|u_j\|^q_{L^p}\right)^{1/q}, \tag{3.69}$$

one point being that in (3.67) the L^p-norm of the function u is dominated by the sum of the L^p-norm of the "coarse grained" projection $E_0 u$ and the second term in (3.67). However, the

key observation here is that, morally, the continuity modulus $\omega_p(u, j)$ is equivalent to some information about the projections u_j at the scale $h = 2^{-j}$.

We will not give the full proof of equivalence here (see Chapter 9 of the book "A Mathematical Introduction to Wavelets" by P. Wojtaszczyk) but prove the following Jackson's inequality that ultimately leads to the equivalence. We assume that

$$|\phi(x)| \le \frac{C}{1+|x|^m}, \tag{3.70}$$

and

$$|\phi'(x)| \le \frac{C}{1+|x|^m}, \tag{3.71}$$

with some $m > 3$.

Proposition 3.9 *There exists a constant $C > 0$ so that for any u with $\omega_p(u, j) < +\infty$ we have*

$$\|u - E_j u\|_p \le C\omega_p(u, j). \tag{3.72}$$

Proof. First, a simple scaling argument shows that it suffices to prove (3.72) only for $j = 0$, and

$$E_0 u(x) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} u(t)\phi(t-k)\phi(x-k)dt,$$

hence

$$u(x) - E_0 u(x) = \int_{-\infty}^{\infty} [u(x) - u(t)]\Phi(t, x)dt,$$

with

$$\Phi(t, x) = \sum_{k\in\mathbb{Z}} \phi(t-k)\phi(x-k).$$

We used here the fact that

$$\int_{-\infty}^{\infty} \Phi(t, x)dt = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \phi(t-k)\phi(x-k)dt = \sum_{k\in\mathbb{Z}} \phi(x-k) \equiv 1.$$

A consequence of (3.70) is that

$$|\Phi(t, x)| \le C \sum_{k\in\mathbb{Z}} \frac{1}{1+|t-k|^m} \cdot \frac{1}{1+|x-k|^m} \le C \int_{-\infty}^{\infty} \frac{1}{1+|t-z|^m} \cdot \frac{1}{1+|x-z|^m} dz$$
$$= C \int_{-\infty}^{\infty} \frac{1}{1+|z|^m} \cdot \frac{1}{1+|x-t-z|^m} dz \le \frac{C}{1+|t-x|^{m-1}}.$$

We get from this:

$$\|u - E_0 u\|_p^p = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} [u(x) - u(t)]\Phi(t, x)dt \right|^p dx \le C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{|u(x) - u(t)|dt}{1+|t-x|^{m-1}} \right)^p dx$$
$$\le C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{|u(x) - u(x+y)|dy}{(1+|y|)^{m-1}} \right)^p dx.$$

We now write $m-1=a+b$ with $a,b\geq 0$ and $ap>p+1$, and $bq>1$, where $1/p+1/q=1$, and us the Hölder inequality in the inner integral:

$$\|u-E_0u\|_p^p\leq C\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{|u(x)-u(x+y)|^p dy}{(1+|y|)^{ap}}\right)\left(\int_{-\infty}^{\infty}\frac{dy}{(1+|y|)^{bq}}\right)^{p/q}dx$$
$$\leq C\int_{-\infty}^{\infty}\frac{1}{(1+|y|)^{ap}}\int_{-\infty}^{\infty}|u(x)-u(x+y)|^p dxdy=C\int_{-\infty}^{\infty}\frac{1}{(1+|y|)^{ap}}\omega_p(f,|y|)^p dy.$$

It is easy to see that

$$\int_{-1}^{1}\frac{1}{(1+|y|)^{ap}}\omega_p(u,|y|)^p dy\leq C\omega_p(y,1)^p,$$

while for the other integral we have

$$\int_{|y|\geq 1}\frac{1}{(1+|y|)^{ap}}\omega_p(u,|y|)^p dy=2\int_{1}^{\infty}\frac{1}{(1+|y|)^{ap}}\omega_p(u,|y|)^p dy.$$

However, there is a simple inequality for the modulus of continuity:

$$\omega_p(u;m\delta)\leq m\omega_p(u;\delta),$$

which we can use in the integral to get

$$\int_{|y|\geq 1}\frac{1}{(1+|y|)^{ap}}\omega_p(u,|y|)^p dy\leq C\int_{1}^{\infty}\frac{y^p}{(1+|y|)^{ap}}\omega_p(u,1)^p dy\leq C\omega_p(y;1)^p,$$

and we are done!

It turns out that the L^p-norms $\|u_j\|_{L^p}$ can be characterized by the wavelet coefficients.

Theorem 3.10 *Assume that the mother wavelet $\psi(x)$ is continuous and $|\psi(x)|\leq C/(1+|x|^{1+\delta})$, for some $\delta>0$. There exist two constants $C_{1,2}$ that depend only on the mother wavelet ψ so that for all j we have (in dimension $n=1$)*

$$C_1\|u_j\|_{L^p(\mathbb{R})}\leq 2^{j(1/2-1/p)}\left(\sum_{k\in\mathbb{Z}}|d_{jk}|^p\right)^{1/p}\leq C_2\|u_j\|_{L^p(\mathbb{R})}. \tag{3.73}$$

Proof. Since this theorem is crucial for understanding the connection between the Besov spaces and the wavelets we will give a detailed proof. First, we claim that, as in the preceding proof, it suffices to consider $j=0$. Indeed, assume that (3.73) is proved for $j=0$. For any j we have

$$u_j(x)=\sum_{k\in\mathbb{Z}}d_{jk}2^{j/2}\psi(2^jx-k),$$

hence the function

$$v_j(x)=2^{-j/2}u_j(2^{-j}x)=\sum_{k\in\mathbb{Z}}d_{jk}\psi(x-k)$$

lies in the space W_0. Applying the claim for $j=0$ to the function v_j (whose W_0 wavelet coefficients are d_{jk}) gives

$$C_1\|v_j\|_{L^p}\leq\left(\sum_{k\in\mathbb{Z}}|d_{jk}|^p\right)^{1/p}\leq C_2\|v_j\|_{L^p}. \tag{3.74}$$

However,

$$\|v_j\|_{L^p}^p = 2^{-jp/2} \int |u_j(2^{-j}x)|^p dx = 2^{(1-p/2)j} \int |u_j(x)|^p dp,$$

so that $\|v_j\|_{L^p} = 2^{j/p-j/2}\|u_j\|_{L^p}$, and (3.74) is nothing but:

$$C_1\|u_j\|_{L^p} \leq 2^{j/2-j/p} \left(\sum_{k\in\mathbb{Z}} |d_{jk}|^p \right)^{1/p} \leq C_2\|u_j\|_{L^p}, \tag{3.75}$$

which is the claim of our theorem for a general j. Hence, we only need to prove the result for $j = 0$. Consider the wavelet coefficients

$$d_{0,k} = \int_{\mathbb{R}} u(x)\psi(x-k)dx$$

and extend them in any way to a continuous function $d(y)$ defined for all $y \in \mathbb{R}$. Then we can write

$$u_0(x) = \sum_{k\in\mathbb{Z}} d_{0,k}\psi(x-k) = \int_{\mathbb{R}} d(y)\psi(x-y)d\mu_y, \tag{3.76}$$

with

$$d\mu_y = \sum_{k\in\mathbb{Z}} \delta(y-k).$$

Hence, for any $g \in L^q(\mathbb{R})$ with $1/p + 1/q = 1$ we have

$$\int_{\mathbb{R}} u_0(x)g(x)dx = \int_{\mathbb{R}\times\mathbb{R}} \psi(x-y)d(y)g(x)dxd\mu_y, \tag{3.77}$$

so that

$$\begin{aligned} \left| \int_{\mathbb{R}} u_0(x)g(x)dx \right| &\leq \int_{\mathbb{R}\times\mathbb{R}} |\psi(x-y)||d(y)||g(x)|dxd\mu_y \qquad (3.78)\\ &\leq \left(\int_{\mathbb{R}\times\mathbb{R}} |d(y)|^p|\psi(x-y)|dxd\mu_y \right)^{1/p} \left(\int_{\mathbb{R}\times\mathbb{R}} |g(x)|^q|\psi(x-y)|dxd\mu_y \right)^{1/q}. \end{aligned}$$

Next, note that

$$\int_{\mathbb{R}} |\psi(x-y)|dx = \|\psi\|_1,$$

and

$$r(x) = \int_{\mathbb{R}} |\psi(x-y)|d\mu_y = \sum_{k\in\mathbb{Z}} |\psi(x-k)|$$

is a bounded periodic continuous function since $\psi(x)$ is continuous and $|\psi(x)| \leq C/(1+|x|^{1+\delta})$. It follows that

$$\left| \int_{\mathbb{R}} u_0(x)g(x)dx \right| \leq C \left(\int_{\mathbb{R}\times\mathbb{R}} |d(y)|^p d\mu_y \right)^{1/p} \left(\int_{\mathbb{R}\times\mathbb{R}} |g(x)|^q dx \right)^{1/q} = C \left(\sum_{k\in\mathbb{Z}} |d_k|^p \right)^{1/p} \|g\|_{L^q}. \tag{3.79}$$

Therefore, recalling the duality between L^p and L^q with $1/p + 1/q = 1$, we conclude that

$$\|u_0\|_{L^p} \leq C \left(\sum_{k \in \mathbb{Z}} |d_k|^p \right)^{1/p}. \tag{3.80}$$

The reverse inequality is proved essentially identically, starting with

$$d_{0,k} = \int_{\mathbb{R}} u_0(x) \psi(x - k) dx,$$

and proceeding as above but this time ending up with

$$\left(\sum_{k \in \mathbb{Z}} |d_k|^p \right)^{1/p} \leq C \|u_0\|_{L^p}. \tag{3.81}$$

3.2 Wavelet shrinkage denoising

The connection between the Besov spaces that explicitly measure the local irregularity of a function and the wavelet coefficients gives a precise connection between the regularity of the function and the size of the wavelet coefficients – functions with small wavelet coefficients for large j are regular. Therefore, suppressing high wavelet coefficients for large j leads to a regularization. The result of Donoho and Johnstone is that such regularization is, in some sense, an optimal way to denoise an image.

Shrinkage of scalars

Before we look at the wavelet shrinkage let us understand some basics of the shrinkage (also known as soft thresholding) for scalars. Given a threshold $\lambda > 0$ we define

$$S_\lambda(t) = \begin{cases} t - \lambda \text{ for } t > \lambda, \\ 0, \text{ for } -\lambda \leq t \leq \lambda, \\ t + \lambda, \text{ for } t < -\lambda. \end{cases} \tag{3.82}$$

Soft thresholding can be viewed from the statistical and variational point of views. Let us first discuss the statistical approach. Assume that the measured signal a_0, which for now we take to be a scalar, consists of a true signal a and noise $n = \sigma w$:

$$a_0 = a + \sigma w. \tag{3.83}$$

Here $\sigma > 0$ measures the amplitude of the noise, and we assume that $-1 \leq w \leq 1$ so that the true signal lies in the interval $(a_0 - \sigma, a_0 + \sigma)$. An estimator is a (deterministic) map S from a_0 to an estimate $\hat{a} = S(a_0)$. We say that the estimator $\hat{a} = S(a)$ satisfies the uniform shrinkage condition if

$$|S(a_0)| \leq |a_0 - \sigma w| \text{ for all } a_0 \text{ and } w \in [-1, 1]. \tag{3.84}$$

Obviously, the shrinkage operator $S_\lambda(a)$ satisfies this condition for all $\lambda \geq \sigma$. The next theorem shows that it is "the best" among all estimators that satisfy the uniform shrinkage condition.

Theorem 3.11 *Among all estimators $\hat{a} = S(a)$ that satisfy the uniform shrinkage condition the shrinkage operator $S_\sigma(a)$ with the threshold σ achieves the minimal shrinkage performance:*

$$|S(a_0)| \leq |S_\sigma(a_0)| \text{ for all } a_0 \in \mathbb{R}, \tag{3.85}$$

and the minimal worst estimation error: for any $a \in \mathbb{R}$ we have

$$\max_{w\in[-1,1]} |S(a+\sigma w) - a| \geq \max_{w\in[-1,1]} |S_\sigma(a+\sigma w) - a|. \tag{3.86}$$

Proof. We may assume without loss of generality that $a \geq 0$. Note that a_0 can be written as

$$a_0 = S_\sigma(a_0) + (a_0 - S_\sigma(a_0)) = S_\sigma(a_0) + \sigma w_0,$$

with $w_0 = (a_0 - S_\sigma(a_0))/\sigma$. We have $-1 \leq w_0 \leq 1$ since

$$|S_\sigma(a_0) - a_0| = \min(|a_0|, \sigma) \leq \sigma.$$

Therefore, for any estimator S that satisfies the uniform shrinkage condition we have

$$|S(a_0)| = |S(S_\sigma(a_0) + \sigma w_0)| \leq |S_\sigma(a_0)|,$$

which is the minimal shrinkage condition (3.85).

It is easy to see that the worst estimation error for the soft thresholding operator $S_\sigma(a)$ occurs when the measurement is

$$a_0^w(a) = a - \min(a, \sigma) = (a - \sigma)_+.$$

In that case we have

$$S_\sigma(a_0^w) = (a - 2\sigma)_+,$$

and thus, for any $a \geq 0$:

$$\max_{w\in[-1,1]} |S_\sigma(a+\sigma w) - a| = |S_\sigma(a_0^w) - a| = \min(a, 2\sigma).$$

On the other hand, for any estimator that satisfies the uniform shrinkage condition we have, using (3.85):

$$\begin{aligned} &\max_{w\in[-1,1]} |S(a+\sigma w) - a| \geq a - |S(a_0^w)| \geq a - |S_\sigma(a_0^w)| = |a - S_\sigma(a_0^w)| \\ &= \max_{w\in[-1,1]} |S_\sigma(a+\sigma w) - a|. \end{aligned} \tag{3.87}$$

This proves (3.86).

The variational approach to shrinkage reformulates the shrinkage operator in terms of a minimization procedure. Given a measurement a_0 consider the functional

$$e_p(t; a_0) = \frac{\lambda}{2}(a_0 - t)^2 + \mu|t|^p. \tag{3.88}$$

When $p = 1$ this functional is philosophically related to the one that appears in the Osher-Rudin algorithm. We define $\hat{a} = S(a_0)$ to be the minimizer of $e_p(t; a_0)$. In (3.88), λ is the parameter responsible for keeping $\hat{a}$ close to a_0 while μ is ultimately responsible for the regularity of the minimizer when we consider non-scalar a.

Theorem 3.12 *The minimizer $\hat{a} = S(a_0)$ of (3.88) exists and is unique. Moreover, $a_0 \cdot \hat{a} \geq 0$.*

Proof. Observe that if $a_0 t < 0$ then

$$e_p(-t; a_0) < e_p(t; a_0),$$

meaning that such t can not be a minimizer of $e_p(t; a_0)$. We may thus assume that $a_0 > 0$ and $t \geq 0$ (the case $a_0 = 0$ is trivial – then $t = 0$ is the unique minimizer). Then we have

$$e_p(t; a_0) = \frac{\lambda}{2}(a_0 - t)^2 + \mu t^p,$$

so that

$$e_p'(t; a_0) = \lambda(t - a_0) + \mu p t^{p-1},$$

and

$$e_p''(t; a_0) = \lambda + \mu p(p-1)t^{p-2} > 0. \tag{3.89}$$

Strict convexity means that the minimizer over $t \geq 0$ is unique, and its existence is obvious since $e_p(t; a_0) \to +\infty$ as $t \to +\infty$.

Expression (3.89) shows that the quadratic term in the definition of $e_p(t; a_0)$ is very important – it guarantees the strict convexity of $e_p(t; a_0)$ even for $p = 1$. The case $p = 1$ is, actually particularly interesting. Then either the minimizer $\hat{a} = 0$ or it satisfies

$$e_p'(\hat{a}; a_0) = 0, \tag{3.90}$$

If $a_0 > 0$ then Theorem 3.12 implies that $\hat{a} \geq 0$, and any non-negative solution of (3.90) (if it exists) satisfies

$$\lambda(t - a_0) + \mu = 0, \quad t \geq 0, \tag{3.91}$$

that is,

$$t = a_0 - \frac{\mu}{\lambda}.$$

We conclude that if $a_0 < \sigma = \mu/\lambda$ then the minimizer is $\hat{a} = 0$ while

$$t = a_0 - \sigma, \quad \text{if } a_0 \geq \sigma. \tag{3.92}$$

The analysis for $a_0 < 0$ is identical. Therefore, we obtain the following result: the minimizer of

$$e_1(t; a_0) = \frac{\lambda}{2}(a_0 - t)^2 + \mu|t| \tag{3.93}$$

is given by the shrinkage operator $\hat{a} = S_\sigma(a_0)$ with $\sigma = \mu/\lambda$.

When $p = 2$, the minimizer satisfies

$$\lambda(t - a_0) + 2\mu t = 0, \tag{3.94}$$

that is, the minimizer is a fixed multiple of a_0:

$$\hat{a} = \frac{\lambda}{\lambda + 2\mu} a_0. \tag{3.95}$$

For $1 < p < 2$ and $a_0 \geq 0$, the minimizer is either $\hat{a} = 0$ or it obeys

$$\lambda(t - a_0) + \mu p t^{p-1} = 0, \tag{3.96}$$

which can not be solved explicitly. On the other hand, an "almost optimizer" is given by the hard truncation: set

$$\sigma = \left(\frac{2\mu}{\lambda}\right)^{1/(2-p)},$$

and

$$\tilde{a} = \begin{cases} 0, \text{ if } |a_0| \leq \sigma, \\ a_0, \text{ if } |a_0| > \sigma. \end{cases} \tag{3.97}$$

Then $\tilde{a}$ is "almost optimal" in the following sense: let us assume again that $a_0 \geq 0$, then for $0 \leq t \leq a_0/2$ we have

$$e_p(t; a_0) \geq \frac{\lambda}{2}(a_0 - \frac{a_0}{2})^2 = \frac{\lambda}{8}a_0^2 = \frac{1}{4}e_p(0; a_0),$$

and for any $t \geq a_0/2$:

$$e_p(t; a_0) \geq \mu\left(\frac{a_0}{2}\right)^p = \frac{1}{2^p}e_p(a_0; a_0),$$

meaning that, as $1 < p < 2$:

$$e_p(t; a_0) \geq \frac{1}{4}\min(e_p(0; a_0), e_p(a_0; a_0)). \tag{3.98}$$

On the other hand, for $\tilde{a}$ we have the following: if $0 \leq a_0 \leq \sigma$ then

$$e_p(\tilde{a}; a_0) = e_p(0; a_0) = \frac{\lambda}{2}a_0^2 \leq \mu(a_0)^p = e_p(a_0; a_0), \tag{3.99}$$

while if $a_0 > \sigma$ we have

$$e_p(\tilde{a}; a_0) = \mu(a_0)^p \leq e_p(0; a_0). \tag{3.100}$$

Therefore, (3.98) says that for all $t \in \mathbb{R}$ we have, whether $|a_0| \geq \sigma$ or not:

$$e_p(t; a_0) \geq \frac{1}{4}e_p(\tilde{a}; a_0), \tag{3.101}$$

thus $\tilde{a}$ given by hard thresholding (3.97) is, indeed, an "almost minimizer" of $e_p(t; a_0)$.

Denoising by wavelet shrinking

The wavelet denoising scheme of Donoho and Johnstone is formulated simply as follows:

$$\hat{u}^* = W^{-1}S_\lambda W(u_0). \tag{3.102}$$

That is, each of the wavelet coefficients d_{jk} of the measured signal

$$u_0(x) = \sum_{j,k\in\mathbb{Z}} d_{jk}\psi_{jk}(x),$$

is soft-thresholded by the operator S_λ:

$$\tilde{d}_{jk} = S_\lambda(d_{jk}),$$

and then the "cleaned-up" signal is reconstructed from the resulting wavelet coefficients:

$$\hat{u}^*(x) = \sum_{j,k} \tilde{d}_{jk}\psi_{jk}(x). \tag{3.103}$$

Note that the mapping $u_0 \to \hat{u}^*$ is non-linear because the shrinkage operator is not linear.

Let us assume that the signal is given by

$$u_0(x) = u(x) + n(x), \tag{3.104}$$

where $u(x)$ is the true signal that we need to recover, and $n(x)$ is noise with variance σ^2. We will consider for simplicity the one-dimensional case and assume that the signal $u \in L^2(\mathbb{R})$ is compactly supported on an interval $[A, B]$.

We will also assume that both the scaling function and the mother wavelet are C^r-smooth and compactly supported on an interval $[a, b]$. It will be convenient to define the domain I_j, for each level $j \geq 0$, of the indices for which the level j wavelets do not vanish on $[A, B]$:

$$I_j = \Big\{(j,k) : \ k \in \mathbb{Z}, \quad \int_A^B |\psi_{jk}(x)|dx > 0\Big\}.$$

Similarly, we set

$$I_{-1} = \Big\{(-1,k) : \ k \in \mathbb{Z}, \quad \int_A^B |\phi(x-k)|dx > 0\Big\}.$$

Each of I_j, $j \geq -1$ is a finite set since both the scaling function ϕ and the mother wavelet ψ are compactly supported. We will also use the notation

$$I_{(J)} = I_{J-1} \cup I_{J-2} \cup \ldots \cup I_0 \cup I_{-1}.$$

We will use the wavelet decomposition in the form slightly different from what we were using so far, writing

$$u(x) = \sum_{j \geq -1, k \in \mathbb{Z}} d_{jk}(u)\psi_{j,k}, \tag{3.105}$$

with the convention that $\psi_{-1,k} = \phi(x-k)$ and

$$d_{-1,k}(u) = \int u(x)\phi(x-k)dx.$$

At each scale $J \geq 0$ we define a subspace F_J of $L^2(\mathbb{R})$ as the set of all functions $u \in L^2(\mathbb{R})$ whose wavelet coefficients d_{jk} vanish unless $(j,k) \in I_j$, and $-1 \leq j < J$. The subspaces F_J form an increasing sequence and their union is $F(A,B)$, the set of all functions $u \in L^2(\mathbb{R})$ whose wavelet coefficients d_{jk} vanish unless $(j,k) \in I_j$, for some $j \geq -1$. Clearly, $F(A,B)$ contains the set $\tilde{F}$ of all functions $u \in L^2(\mathbb{R})$ supported inside the interval $[A,B]$.

The spaces $F_J(A,B)$ and $F(A,B)$ satisfy the following solid and orthosymmetric condition introduced by Donoho: if d_{jk} are the wavelet coefficients of a function

$$u(x) = \sum_{j\geq -1, k\in\mathbb{Z}} d_{jk}(u)\psi_{j,k}$$

that lies in F_J or F, and the sequence s_{jk} satisfies $|s_{jk}| \leq 1$ for all $j \geq -1$, $k \in \mathbb{Z}$, then the function

$$u(x) = \sum_{j\geq -1, k\in\mathbb{Z}} s_{jk} d_{jk}(u)\psi_{j,k}$$

also lies, respectively, in F_J or F. This property makes the spaces F_J and F much more amenable to analysis than the space $\tilde{F}$, and this is what we will use. The following theorem counts the number of indices in $I_{(J)}$.

Theorem 3.13 *Assume that the functions ϕ and ψ have the same support $[a,b]$ and $l = b-a$. Define also $L = B - A$, $M = \#I_{(J)}$ and $N = 2^J$, then*

$$NL + (l-1)(\log N + 1) \leq M < NL + (l+1)(\log N + 1). \tag{3.106}$$

Proof. It follows from the definition of I_j that $(j,k) \in I_j$ for $j \geq 0$ if and only if the sets $a < 2^j x - k < b$ and $A < x < B$ have a non-empty intersection, that is, the intervals (a,b) and $(2^j A - k, 2^j B - k)$ overlap. It is straightforward to compute then that

$$2^j L + l - 1 \leq \#I_j < 2^j L + l + 1, \quad j \geq 0.$$

On the other hand, I_{-1} obeys the same estimate as I_0 since the support of ϕ is the same as that of ψ, hence

$$L + (l-1) + \sum_{j=0}^{J-1}(2^j L + l - 1) \leq M < L + (l-1) + \sum_{j=0}^{J-1}(2^j L + l + 1),$$

that is,

$$2^J L + (l-1)(J+1) \leq M < 2^J L + (l+1)(J+1),$$

which is exactly what we need.

We will need the following estimate for the growth of a sequence of Gaussian independent identically distributed random variables.

Theorem 3.14 *Let z_k be a sequence of i.i.id Gaussian random variables with mean zero and variance σ^2, then*

$$p_M = Prob\left(\max_{1\leq k\leq M} |z_k| \leq \sigma\sqrt{2\log M}\right) \to 1, \ as\ M \to +\infty. \tag{3.107}$$

Let us now go back to the problem of denoising functions from F_J. Taking the wavelet transform of (3.104) we get

$$d^0_{jk} = d_{jk} + z_{jk}, \quad j \geq -1, \ k \in \mathbb{Z}. \tag{3.108}$$

Here, d^0_{jk} are the wavelet coefficients of the recorded signal $u_0(x)$, d_{jk} are those of the original signal $u(x)$ and z_{jk} are the wavelet coefficients of the noise $n(x)$. The problem is to recover d_{jk} from the noisy measurement d^0_{jk}.

In order to address this problem we need to adopt a model for the noise $n(x)$. We will assume that this is a Gaussian white noise with variance σ^2. This means that $n(x)$ is a random distribution so that for each deterministic test function $\phi \in C_c^\infty$ the value $\langle n, \phi\rangle$ is a Gaussian random variable of mean zero and with variance $\sigma^2\|\phi\|^2_{L^2}$. More generally, for any pair of deterministic test functions $\phi, \psi \in C_c^\infty$ the random variables $\langle n, \phi\rangle$ are jointly Gaussian random variables with mean zero and covariance

$$\mathbb{E}[\langle n, \phi\rangle\langle n, \psi\rangle] = \sigma^2 \int \psi(x)\phi(x)dx.$$

Formally, this fact may be expressed as follows: $\mathbb{E}(n(x)) = 0$ and

$$\mathbb{E}(n(x)n(y)) = \delta(x - y).$$

The wavelet coefficients for the Gaussian white noise are well defined if the wavelets $\psi_{jk}(x)$ are rapidly decaying or compactly supported:

$$z_{jk} = \langle n, \psi_{jk}\rangle,$$

and are mean zero Gaussian random variables with the covariance

$$\mathbb{E}[z_{jk}z_{j'k'}] = \mathbb{E}[\langle n, \psi_{jk}\rangle\langle n, \psi_{j'k'}\rangle] = \sigma^2\langle\psi_{jk}, \psi_{j'k'}\rangle = \sigma^2\delta_{jj'}\delta_{kk'}. \tag{3.109}$$

Therefore, the wavelet coefficients z_{jk} are independent identically distributed Gaussian random variables with mean zero and variance σ^2.

Theorem 3.14 indicates that a reasonable cut-off in the soft thresholding of wavelet coefficients is $\lambda = \sigma\sqrt{2\log M}$ (recall that M is the number of elements in $I_{(J)}$). Note that according to Theorem 3.13 we have

$$\log M \approx \log N + \log L,$$

so that the threshold is

$$\lambda \approx \sigma\sqrt{2(\log N + \log L)}. \tag{3.110}$$

In particular, if the support of the function $u(x)$ is not too large: $L \ll N$, then $\lambda \approx \sigma\sqrt{2\log N}$. On the other hand, if $N \sim L$ then we need to take into account both terms in expression (3.110).

We will define the shrinkage as follows. First, for any index $(j, k) \notin I_{(J)}$ we set

$$\hat{d}^*_{jk} = 0. \tag{3.111}$$

For the wavelet coefficients with $(j, k) \in I_{(J)}$ we set

$$\hat{d}^*_{jk} = S_\lambda(d^0_{jk}). \tag{3.112}$$

Then the denoised image is the inverse wavelet transform of $\hat{d}^*_{jk}$:

$$\hat{u}^* = W^{-1}(\hat{d}^*_{jk}, \quad j \geq 1, \ k \in \mathbb{Z}). \tag{3.113}$$

Theorem 3.15 *Given any $u \in F_J(A,B)$, with probability at least p_M given by (3.107) the estimator $\hat{u}^*$ defined by (3.113) has a smaller norm in the Besov space $B^\alpha_{p,p}(\mathbb{R})$ than the original image u, that is,*

$$Prob\left(\|\hat{u}^*\|_{B^\alpha_{p,p}(\mathbb{R})} \leq \|u\|_{B^\alpha_{p,p}(\mathbb{R})}\right) \geq p_M. \tag{3.114}$$

Here the Besov norm is defined in terms of the wavelet coefficients:

$$\|u\|_{B^\alpha_{p,p}(\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} |d_{-1,k}|^p\right)^{1/p} + \left(\sum_{j\geq 0, k\in\mathbb{Z}} 2^{jp(\alpha+1/2-1/p)}|d_{jk}|^p\right)^{1/p}.$$

An immediate corollary of Theorems 3.14 and 3.15 is that

$$\text{Prob}\left(\|\hat{u}^*\|_{B^\alpha_{p,p}(\mathbb{R})} \geq \|u\|_{B^\alpha_{p,p}(\mathbb{R})}\right) \to 0 \text{ as } M \to +\infty. \tag{3.115}$$

Proof. Let $u \in B^\alpha_{p,p}(\mathbb{R}) \cap F_J(A,B)$. First, we have $\hat{d}^*_{jk} = 0$ for all $(j,k) \notin I_{(J)}$. Second, if $|z_{jk}| \leq \lambda$ we have

$$|\hat{d}^*_{jk}| \leq |d_{jk}|, \quad (j,k) \in I_{(J)}.$$

On the other hand, we have $|z_{jk}| \leq \lambda$, with probability p_M, so the conclusion of the theorem follows.

Let us now discuss the optimality of wavelet shrinkage among all estimators which act component-wise on the wavelet coefficients, that is $d_{jk}(\hat{u})$ depend only on d^0_{jk}, all $j \geq -1$, $k \in \mathbb{Z}$. We will also assume that $\hat{d}_{jk} = 0$ for $(j,k) \notin I_{(J)}$. The mean squared estimation error is

$$e_J(u;\hat{u}) = \mathbb{E}\|\hat{u} - u\|^2_{L^2},$$

and its worst estimation error is

$$e_J(\hat{u}) = \sup_{u\in F_J(A,B)} e_J(u;\hat{u}).$$

Finally, the best performance e^*_J is the infimum of $e_J(\hat{u})$ over all estimators $\hat{u}$. The shrinkage estimator $\hat{u}^*$ is nearly optimal in the following sense (this is an important result of Donoho and Johnstone):

$$e_J(u;\hat{u}^*) \leq (2\log M + 1)(\sigma^2 + 2.22e^*_J), \text{ for any } u \in F_J(A,B), \tag{3.116}$$

where $M = \#I_{(J)}$ and σ^2 is the noise variance.

The variational approach to wavelet shrinkage

Wavelet shrinkage can also be formulated as a variational problem, similar to scalar shrinkage. Given an observation u_0, consider the following cost functional:

$$E(u;u_0) = \|u\|^p_{B^\alpha_{p,p}} + \frac{\lambda}{2}\int_{\mathbb{R}} (u_0(x) - u(x))^2 dx. \tag{3.117}$$

Here, the Besov norm is defined in terms of the wavelet coefficients (we consider for simplicity the one dimensional case):

$$\|u\|^p_{B^\alpha_{p,p}} = \sum_{j\geq -1, k\in\mathbb{Z}} 2^{jp(\alpha+1/2-1/p)}|d_{jk}|^p. \tag{3.118}$$

The denoised image $\hat{u}$ is the

$$\hat{u} = \text{argmin } E(u; u_0).$$

Note that $E(u; u_0)$ has the form

$$E(u; u_0) = \sum_{j\geq -1, k\in\mathbb{Z}} 2^{jp(\alpha+1/2-1/p)}|d_{jk}|^p + \frac{\lambda}{2}\sum_{j\geq -1, k\in\mathbb{Z}} (d_{jk} - d^0_{jk})^2, \tag{3.119}$$

and thus the minimization problem is completely decoupled in the wavelet domain. That is, we have

$$\hat{d}_{jk} = \text{argmin } e_{jk}(d_{jk}; d^0_{jk}), \tag{3.120}$$

with

$$e_{jk}(t; d^0_{jk}) = \mu_j|t|^p + \frac{\lambda}{2}(t - d^0_{jk})^2,$$

and $\mu_j = 2^{jp(\alpha+1/2-1/p)}$. This is the scalar shrinkage problem we have analyzed above and we have the following.

Theorem 3.16 *In one dimension the wavelet coefficients of the denoised image $\hat{u}$ for the cost function $E(u; u_0)$ with $p = 1$ are given explicitly by the shrinkage operator*

$$\hat{d}_{jk} = S_{\sigma_j}(d^0_{jk}) = sgn(d^0_{jk})(d^0_{jk} - \sigma_j)_+, \tag{3.121}$$

with $\sigma_j = \mu_j/\lambda$.

Note that if $\alpha = 1/2$ then the thresholds $\sigma_j = 1/\lambda$ are independent of j – this gives uniform shrinkage.

As we did for shrinkage of scalars, when $1 < p < 2$ we define the threshold

$$\sigma_j = \left(\frac{2\mu_j}{\lambda}\right)^{1/(2-p)},$$

and consider the hard thresholding estimator

$$\hat{d}_{jk} = d^0_{jk}H(|d^0_{jk}| - \sigma_j),$$

where $H(t)$ is the Heaviside function. Then the estimator

$$\hat{u} = W^{-1}(\hat{d}_{jk},\ j \geq -1, k \in \mathbb{Z})$$

is nearly optimal for the cost function $E_p(u; u_0)$ in the sense that

$$E_p(\hat{u}; u_0) \leq 4 \min E(u; u_0).$$

Therefore, as for scalar shrinkage, wavelet shrinkage can be understood both from the statistical and variational points of view.

4 Image deblurring

We will consider three types of blur that have different physical origins and should be corrected differently. First, there is optical blur that appears if the lens of an imaging device is not properly focused on the subject – this is a common problem in everyday photography. Sometimes it is unavoidable if there are many subjects in the image that are various distances from the lens so that they can not simultaneously be in focus, especially if a small aperture may not be used due to low light. Motion blur is also a common occurrence in photography and has two main sources: the subject may move during the exposure as in the photograph of a dancing child or of a rapidly moving car, especially if the movement is in the direction parallel to the sensor plane. Another source of motion blur is the movement of the camera itself, for instance, if the exposure is long and the camera is not on a tripod, or the tripod is of poor quality. The third type of blur is the atmospheric blur that comes from light scattering between the object and the lens due to small particles in the atmosphere. A common example are photographs taken in the fog.

4.1 Mathematical models of blur

Motion blur

Let us assume that the object is moving at speed $v \in \mathbb{R}^2$ during the exposure that takes time T. This means that the image at each point x will be an integral of the original image over all points in the object over the interval that connects the point x and the point $x - vT$: the recorded image u_0 is related to the unblurred image $u(x)$ by

$$u(x) = \int_0^T u(x - vt)dt. \tag{4.1}$$

This can be written as

$$u(x) = \int_{\mathbb{R}^2} k(x - y)u(y)dy, \tag{4.2}$$

with the convolution kernel

$$k(x) = \int_0^T \delta(x - vt)dt. \tag{4.3}$$

It is, of course, possible that the object is moving at a non-uniform speed. Another possibility is that the intensity of the recording changes during the exposure. This means that the kernel $k(x, y)$ does not have the simple form above, but more generally we have

$$u(x) = \int_{\mathbb{R}^2} k(x, y)u(y)dy, \tag{4.4}$$

where the kernel $k(x, y)$ is supported on the set of y that reach the point x during the recording, and the magnitude of $k(x, y)$ corresponds to the level of recording at the time the point y of the object passes the point x in the image domain.

Models for the out of focus blur

Out of focus blur occurs when the lens is not properly focused on the subject. This means that the recorded image $u_0(x)$ is the average of the original image $u(x)$ over a disk $B(x;r)$ of radius r around the point x:

$$u_0(x) = \frac{1}{\pi r^2} \int_{B(x,r)} u(y)dy. \tag{4.5}$$

Here the radius r depends on how out of focus the image is – the in focus image corresponds to $r = 0$ and the larger r the more out of focus the image is. This may also be written as a convolution:

$$u_0(x) = \int_{\mathbb{R}} k(x-y)u(y)dy, \tag{4.6}$$

with

$$k(x) = \frac{1}{\pi r^2} \chi_{|x|\leq r}(x). \tag{4.7}$$

It is also possible that the defocusing varies smoothly with the radius, so the kernel $k(x)$ may have more general form that (4.7) such as the Gaussian kernel

$$k(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{|x|^2}{2\sigma^2}\right). \tag{4.8}$$

Models of the atmospheric blur

One reason for the atmospheric blur is multiple scattering of light by minuscule particles in the air. A characteristic feature of this phenomenon is that light at various frequencies scatters at different levels – usually the shorter the wave length the stronger the scattering. This is one reason why the sun and the moon are so yellow when they are low above the horizon – at such time the distance the light travels through the atmosphere is the longest, hence the difference in scattering of blue and yellow light is the strongest. The recorded image is then best related to the original image in the Fourier domain:

$$\hat{u}_0(\xi) = K(\xi)u(\xi), \quad \xi \in \mathbb{R}^2. \tag{4.9}$$

Here $K(\xi)$ is the damping factor for wave vector ξ. Therefore, in the physical domain we have a convolution:

$$u_0(x) = \int_{\mathbb{R}^2} k(x-y)u(y)dy, \tag{4.10}$$

where $k(x)$ is the Fourier transform of the function $K(\xi)$.

General blur models

In all three examples above, the recorded signal $u_0(x)$ is a convolution of the original image $u(x)$ with a kernel $k(x)$:

$$u_0(x) = \int_{\mathbb{R}^2} k(x-y)u(y)dy. \tag{4.11}$$

More generally, if the blur is not shift-invariant and various parts of the image are blurred in a different way, we have

$$u_0(x) = \int_{\mathbb{R}^2} k(x,y)u(y)dy. \tag{4.12}$$

Abstractly, we can think of blur as a linear operator $K : \ L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^q)$, with some $1 \le p, q \le +\infty$. This operator should satisfy the condition

$$K[1] = 1, \tag{4.13}$$

where 1 denotes a function that is identically equal to one. For translation-invariant operators of the form (4.11) this is equivalent to

$$\int_{\mathbb{R}^2} k(x)dx = 1, \tag{4.14}$$

while for the operators of the form (4.12) this is equivalent to

$$\int_{\mathbb{R}^2} k(x,y)dy = 1, \text{ for all } x \in \mathbb{R}^2. \tag{4.15}$$

The function $k(x)$ (or $k(x,y)$) is known as the point spread function (PSF). Typically, blur operators are smoothing meaning that $K[u]$ is more regular than the function u. In terms of the convolution kernels $k(x-y)$ this means that the Fourier transform $K(\xi) = \hat{k}(\xi)$ decays rapidly in ξ, while for general kernels $k(x,y)$ this translates into good smoothness properties of $k(x,y)$ in x.

Basic difficulties of deblurring

Deblurring problems are ill-posed, as can be seen from several perspectives. Inverting an operator of the form

$$u_0(x) = K[u](x) = \int_{\mathbb{R}^2} k(x-y)u(y)dy$$

is formally deceptively easy: all we need to do is write

$$\hat{u}(\xi) = \frac{\hat{u}_0(\xi)}{K(\xi)}, \tag{4.16}$$

where $K(\xi)$ is the Fourier transform for $k(x)$. However, typically $k(x)$ is a smooth function meaning that $K(\xi)$ decays rapidly in ξ. Therefore, the naive division in (4.16) magnifies the measurement errors for large ξ dramatically which is not what we want or need.

Another way to look at the ill-posedness of deblurring is to realize that since the blur operator is usually regularizing, it is often a compact operator from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$. Inverting compact operators is typically ill-posed. Qualitatively this is because they are very close to finite-rank operators which can not be invertible since they map an infinite-dimensional space onto a finite-dimensional space. Therefore, compact operators are well approximated by non-invertible operators, which should obviously cause computational problems in inversion.

Therefore, it is essential to regularize the deblurring problem to make it well-posed – this is usually done by considering variational problems with additional regularizing terms. This is what we will study next.

4.2 Deblurring BV images with a known PSF

The variational formulation

We will now consider the blur models that are given by a convolution and additive noise:

$$u_0(x) = k \star u + n(x) = \int_{\mathbb{R}^2} k(x-y)u(y)dy + n(x), \tag{4.17}$$

with a known PSF $k(x)$. Here $u_0(x)$ is our measurement, $u(x)$ is the original image we need to recover and $n(x)$ is an additive noise. We will look for the function $u(x) \in BV(\mathbb{R}^2)$ as a minimizer of the energy functional

$$E(u; u_0) = \alpha \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (k \star u - u_0)^2 dx, \tag{4.18}$$

with $\Omega = \mathbb{R}^2$. This model reduces to the Osher-Rudin model that we have already studied when there is no blur: $k(x) = \delta(x)$. The parameters α and λ are responsible for how much weight we put on the regularity of u and how much on the faithfulness to the measurement. In principle, only the ratio $r = \alpha/\lambda$ matters for the minimization problem – $r \gg 1$ means that smoothness is very important while $r \ll 1$ means that smoothness is less important than the requirement that $u_0 \approx k \star u$.

As in practice imaging is done in bounded domains, it is instructive to pass from $\Omega = \mathbb{R}^2$ to a bounded domain Ω. The blur operator is then given by

$$K[u](x) = \int_\Omega k(x,y)u(y)dy. \tag{4.19}$$

There are several ways to pass to the kernel $k(x,y)$ from the shift-invariant kernel $k(x-y)$ (shift-invariant kernels may only work for $\mathbb{R}^2$). The simplest way is to define

$$k(x,y) = \frac{k(x-y)}{\int_\Omega k(x-z)dz}, \quad \text{for } x, y \in \Omega. \tag{4.20}$$

Then the blur operator satisfies $K[1] = 1$.

Another approach consists of extending the function $u(y)$ in some way to the whole space and then applying the convolution with the PSF $k(x)$. One way is to set

$$\tilde{u}(y) = \int_\Omega g(y,z)u(z)dz, \quad y \in \mathbb{R}^2,$$

with some kernel $g(y,z)$, $y \in \mathbb{R}^2$, $z \in \Omega$ and set

$$K[u] = \int_{\mathbb{R}^2} k(x-y)\tilde{u}(y)dy = \int_{\mathbb{R}^2 \times \Omega} k(x-y)g(y,z)u(z)dzdy. \tag{4.21}$$

In order to satisfy the condition $K[1] = 1$, the function $g(y,z)$ should obey

$$\int_{\mathbb{R}^2 \times \Omega} k(x-y)g(y,z)dzdy = 1 \text{ for all } x \in \Omega. \tag{4.22}$$

Any such approach eventually leads to the variational problem of minimizing the energy

$$E(u; u_0) = \alpha \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (K[u] - u_0)^2 dx. \tag{4.23}$$

Existence and uniqueness of the minimizer

In order to study existence and uniqueness of the minimizer of the energy functional (4.23) we will assume that the blur operator $K: \ L^1(\Omega) \to L^2(\Omega)$ is linear, bounded and injective, and satisfies the condition $K[1] = 1$.

Theorem 4.1 *Assume the above conditions on the blur operator K and that $u_0 \in L^2$. Then the minimizer of the energy functional (4.23) in $BV(\Omega)$ exists and is unique.*

Proof. The total variation is first-order homogeneous in u, while the functional

$$\tilde{E}(u; u_0) = \int_\Omega (K[u] - u_0)^2 dx$$

is convex. Since $K[u]$ is injective, it is strictly convex. Indeed, we have, for any $u, v \in \mathrm{BV}(\Omega)$ and $\lambda \in (0, 1)$:

$$\begin{aligned}
&\tilde{E}(\lambda u + (1-\lambda)v; u_0) = \int_\Omega (\lambda K[u] + (1-\lambda)K[v] - u_0)^2 dx \\
&= \int_\Omega (\lambda(K[u] - u_0) + (1-\lambda)(K[v] - u_0))^2 dx \\
&\leq \lambda \int_\Omega (K[u] - u_0)^2 dx + (1-\lambda) \int_\Omega (K[v] - u_0)^2 dx = \lambda \tilde{E}(u; u_0) + (1-\lambda)\tilde{E}(v; u_0)
\end{aligned}$$

with equality possible only if $K[u] = K[v]$ which is impossible unless $u = v$ since K is injective. Therefore, the functional $E(u; u_0)$ is strictly convex, and the minimizer, if it exists, is unique.

In order to prove existence of a minimizer, let u_n be a minimizing sequence for $E(u; u_0)$ – such sequence exists since $E(u; u_0)$ is bounded from below (it is positive). Then $E(u_n; u_0)$ is uniformly bounded from above, and hence so is the total variation

$$\mathrm{TV}(u_n) = \int_\Omega |Du_n|.$$

We have the Poincaré inequality

$$\int_\Omega |u_n - \langle u_n \rangle| dx \leq C_p(\Omega) \int_\Omega |Du_n|,$$

where

$$\langle u \rangle = \frac{1}{|\Omega|} \int_\Omega u dx.$$

Hence, the sequence $g_n = u_n - \langle u_n \rangle$ is uniformly bounded in $L^1(\Omega)$. Therefore, as the operator K is bounded from $L^1(\Omega) \to L^2(\Omega)$, and since $K[1] = 1$, the sequence

$$K[g_n] = K[u_n] - \langle u_n \rangle$$

is bounded in $L^2(\Omega)$. But since $K[u_n]$ is bounded in $L^2(\Omega)$ (because $E(u_n; u_0)$ is bounded from above), the sequence $\langle u_n \rangle$ must be bounded as well. The Poincaré inequality then implies that u_n is bounded in $L^1(\Omega)$ whence in $BV(\Omega)$ (we already know that $\mathrm{TV}(u_n)$ is uniformly

bounded). Therefore, there exists a subsequence u_{n_k} that converges to a function u_* in $L^1(\Omega)$ and such that

$$\int_\Omega |Du_*| \leq \liminf_{k\to\infty} \int_\Omega |Du_{n_k}|.$$

Moreover, as K is continuous from $L^1(\Omega) \to L^2(\Omega)$, we know that

$$\int_\Omega (K[u_*] - u_0)^2 dx = \lim_{k\to\infty} \int_\Omega (K[u_{n_k}] - u_0)^2 dx.$$

Therefore, we have

$$E(u_*; u_0) \leq \liminf_{k\to+\infty} E(u_n; u_0),$$

and, as u_n is a minimizing sequence, u_* is a minimizer, and we are done.

Let us now show that the minimizer satisfies the constraint

$$\langle K[u_*] \rangle = \langle u_0 \rangle. \tag{4.24}$$

Let us define, for $c \in \mathbb{R}$:

$$e(c) = E(u_* - c; u_0).$$

Then, as $K[1] = 1$ and $TV(u_* - c) = TV(u_*)$, the minimizer c_* of $e(c)$ should minimize

$$\tilde{e}(c) = \int_\Omega (K[u_*] - c - u_0)^2 dx.$$

Hence, it satisfies

$$2c_*|\Omega| - 2\int_\Omega (K[u_*] - u_0) dx = 0. \tag{4.25}$$

On the other hand, since u_* is the minimizer of $E(u; u_0)$ we must have $c_* = 0$, and (4.24) follows from (4.25).

4.3 Variational deblurring with unknown PSF

We will consider two cases when the point spread function is unknown: first, when it is not known but belongs to a certain set parametrized by $\theta \in \mathbb{R}^d$, and, second, when we do not know the PSF at all, which is a completely blind deblurring.

Parametric blind deblurring

Let us first assume that the unknown PSF belongs to a parametric family

$$\mathcal{K} = \{K_\theta : \ \theta \in I \subseteq \mathbb{R}^d\}.$$

A typical example would be the Gaussian family

$$g(x; \theta) = \frac{1}{2\pi\theta} \exp\left(-\frac{x_1^2 + x_2^2}{2\theta}\right),$$

with $\theta \in I = (0, +\infty)$.

Then, in order to incorporate the parametric dependence into the variational problem we introduce a cost function $\phi(\theta)$ associated to the kernel K_θ. The deblurring minimization problem is to minimize, over $u \in \mathrm{BV}(\Omega)$ and $\theta \in I$, the functional

$$E(u, \theta; u_0) = \alpha \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (K_\theta[u] - u_0)^2 dx + \phi(\theta). \tag{4.26}$$

We will assume that the function $\phi(\theta)$ is uniformly bounded from below: $\phi(\theta) \geq -M > -\infty$ for all $\theta \in I$. We will also assume that, for each $\theta \in I$ fixed, the operator K_θ is a blur operator: it is injective, bounded from $L^1(\Omega)$ to $L^2(\Omega)$, and $K_\theta[1] = 1$. Then for each $\theta \in I$ we can find the unique minimizer $\hat{u}_\theta$ of the functional

$$\tilde{E}(u; \theta, u_0) = \alpha \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (K_\theta[u] - u_0)^2 dx, \tag{4.27}$$

with θ fixed. The problem of minimizing $E(u, \theta; u_0)$ is thus reduced to minimizing, over $\theta \in I$, the function

$$e(\theta; \hat{u}_\theta, u_0) = \tilde{E}(\hat{u}_\theta; \theta, u_0). \tag{4.28}$$

This leads to an approach to minimizing $E(u, \theta; u_0)$ known as alternating minimization. It is defined as follows. Given an initial guess $\theta^{(0)}$ we set

$$u^{(0)} = \text{argmin } \tilde{E}(u; \theta^{(0)}, u_0), \tag{4.29}$$

and then update θ to

$$\theta^{(1)} = \text{argmin } e_1(\theta; u^{(0)}, u_0) \tag{4.30}$$

with

$$e_1(\theta; v, u_0) = \frac{\lambda}{2} \int_\Omega (K_\theta[v] - u_0)^2 dx + \phi(\theta). \tag{4.31}$$

Then we proceed iteratively:

$$u^{(n)} = \text{argmin } \tilde{E}(u; \theta^{(n)}, u_0), \tag{4.32}$$

and

$$\theta^{(n+1)} = \text{argmin } e_1(\theta; u^{(n)}, u_0) \tag{4.33}$$

A simple observation is that energy is decreasing in n:

$$E(u^{(n+1)}, \theta^{(n+1)}; u_0) \leq E(u^{(n)}, \theta^{(n)}; u_0). \tag{4.34}$$

This is because

$$E(u^{(n+1)}, \theta^{(n+1)}; u_0) \leq \tilde{E}(u^{(n)}; \theta^{(n+1)}, u_0) + \phi(\theta^{(n+1)}) = \alpha \int_\Omega |Du^{(n)}| + e_1(\theta^{(n+1)}; u^{(n)}, u_0)$$
$$\leq \alpha \int_\Omega |Du^{(n)}| + e_1(\theta^{(n)}; u^{(n)}, u_0) = E(u^{(n)}, \theta^{(n)}; u_0).$$

We used the definition of $u^{(n+1)}$ in the first inequality above, and the definition of $\theta^{(n+1)}$ in the second inequality, as minimizers of respective functionals.

We have then the following conditional convergence theorem.

Theorem 4.2 *Assume that the map $\theta \to K_\theta$ is continuous from $I \subset \mathbb{R}^n$ to the space $\mathcal{B}(L^1, L^2)$ of bounded linear operators from L^1 to L^2. Suppose also that the function $\phi(\theta)$ is lower semi-continuous. If $u^{(n)} \to u_*$ in $L^1(\Omega)$ and $\theta^{(n)} \to \theta_* \in I$ then the pair (u_*, θ_*) satisfies*

$$u_* = \textit{argmin } \tilde{E}(u; \theta_*, u_0) \tag{4.35}$$

$$\theta_* = \textit{argmin } e_1(\theta; u_*, u_0). \tag{4.36}$$

Proof. Let us first show that

$$K_{\theta^{(n)}}[u^{(n)}] \to K_{\theta_*}[u_*] \text{ in } L^2(\Omega). \tag{4.37}$$

To see that, we write

$$\begin{aligned} \|K_\theta[u] - K_{\theta_*}[u_*]\|_2 &\le \|K_\theta[u] - K_{\theta_*}[u]\|_2 + \|K_{\theta_*}[u] - K_{\theta_*}[u_*]\|_2 \\ &\le \|K_\theta - K_{\theta_*}\|_{L^1 \to L^2} \|u\|_1 + \|K_{\theta_*}\|_{L^1 \to L^2} \|u - u_*\|_1. \end{aligned}$$

Therefore, if $u^{(n)} \to u_*$ in L^1 so that $\|u - u_*\|_1 \to 0$, and $\theta^{(n)} \to \theta_*$ so that

$$\|K_{\theta^{(n)}} - K_{\theta_*}\|_{L^1 \to L^2} \to 0,$$

we have (4.37).

By construction, we have

$$\tilde{E}(u^{(n)}; \theta^{(n)}, u_0) \le \tilde{E}(u; \theta^{(n)}, u_0),$$

for all $u \in \mathrm{BV}(\Omega)$, or, equivalently:

$$\alpha \int_\Omega |Du^{(n)}| + \frac{\lambda}{2} \|K_{\theta^{(n)}}[u^{(n)}] - u_0\|_2^2 \le \alpha \int_\Omega |Du| + \frac{\lambda}{2} \|K_{\theta^{(n)}}[u] - u_0\|_2^2.$$

Passing to the limit $n \to +\infty$ gives, using (4.37):

$$\alpha \int_\Omega |Du_*| + \frac{\lambda}{2} \|K_{\theta_*}[u_*] - u_0\|_2^2 \le \alpha \int_\Omega |Du| + \frac{\lambda}{2} \|K_{\theta_*}[u] - u_0\|_2^2, \text{ for all } u \in \mathrm{BV}(\Omega).$$

This proves (4.35). The proof of (4.36) is basically identical.

Non-parametric blind deblurring

Let us now assume that we do not know anything about the blur operator K except that it satisfies the condition $K[1] = 1$ and some extra assumptions to be specified later. We will consider only the case when the domain Ω is $\mathbb{R}^2$, and the blur is given by a convolution

$$K[u](x) = \int_{\mathbb{R}^2} k(x - y) u(y) dy,$$

with an unknown function $k(x)$. Given a measurement $u_0(x)$, the goal is to recover the original image $u(x)$, and, as a by-product, the kernel $k(x)$.

As usual, we will do this via energy minimization. The energy functional will be of the form

$$E(u,k;u_0) = \alpha \int_\Omega |Du| + \frac{\lambda}{2}\int_{\mathbb{R}^2} (k \star u - u_0)^2 dx + E(k), \tag{4.38}$$

and the key aspect is to devise a good energy $E(k)$.

Many blurs such as the motion blur and the out-of-focus blur involve PSF's $k(x)$ of compact support and with sharp boundaries. For such kernels, the BV norm is an appropriate energy, taking

$$E[k] = \beta \int_{\mathbb{R}^2} |Dk|.$$

Then the total energy is

$$E(u,k;u_0) = \alpha \int_\Omega |Du| + \frac{\lambda}{2}\int_{\mathbb{R}^2} (k \star u - u_0)^2 dx + \beta \int_{\mathbb{R}^2} |Dk|. \tag{4.39}$$

We stress that the fact that we use the BV energy both for $u(x)$ and $k(x)$ is somewhat of coincidence – that is, the BV energy is suitable for $u(x)$ and $k(x)$ different reasons. For instance, if we were expecting $k(x)$ to be a smooth Gaussian-like kernel we would not use the BV-norm as the energy of $k(x)$.

Extra care should be taken when working in the whole space as many compactness theorems are false. For example, a uniform bound $\|u_n\|_{BV(\mathbb{R}^2)} \leq C$ does not imply that u_n has a subsequence converging in $L^1(\mathbb{R}^2)$ (unlike in a bounded domain). A simple counterexample is provided by a sequence $u_n(x) = \phi(x_1 - n, x_2)$ with a smooth compactly supported function $\phi(x)$. This sequence is obviously uniformly bounded in $BV(\mathbb{R}^2)$ but it can not converge strongly in L^1 since it converges weakly to zero but its L^1-norm does not tend to zero as $n \to +\infty$.

As a technical tool, we will need the following form of the Poincaré inequality.

Theorem 4.3 *(Poincaré inequality for $BV(\mathbb{R}^2)$.) There exists a constant C so that for all $u \in BV(\mathbb{R}^2)$ with*

$$\|u\|_{BV(\mathbb{R}^2)} = \|u\|_{L^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} |Du| < +\infty$$

we have

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C \int_{\mathbb{R}^2} |Du|. \tag{4.40}$$

Proof. The standard Sobolev embedding implies that for any ball B_R we have

$$\|u - \langle u \rangle\|_{L^q(B_R)} \leq C_R \|Du\|_{L^p(B_R)},$$

for each $1 \leq p < n$ and all $1 \leq q \leq p^* = np/(n-p)$. Here we denote by

$$\langle u \rangle = \frac{1}{|B_R|}\int_{B_R} u(x)dx.$$

In two dimensions we have $n = 2$ so for $p = 1$ the critical $p^* = 2$. That is, there exists a constant $C_p(R)$ so that for any disk B_R and any $u \in BV(B_R)$ we have

$$\|u - \langle u \rangle\|_{L^2(B_R)} \leq C_p(R) \int_{B_R} |Du|. \tag{4.41}$$

A simple scaling argument shows that the constant C_p does not depend on R: the heuristic reason is that both the left and the right sides in (4.41) have the dimension of $[R]$, hence their ratio does not depend on R (this is easy to make rigorous by looking at a rescaled function $u_R(x) = u(Rx)$ defined in the unit ball B_1). It follows that

$$\|u\|_{L^2(B_R)} \le C \int_{B_R} |Du| + |\langle u \rangle| |B_R|^{1/2} \le C \left[\int_{B_R} |Du| + \frac{\|u\|_{L^1}}{|B_R|^{1/2}} \right]$$

Letting $R \to +\infty$ we get (4.40) provided that $u \in L^1(\mathbb{R}^2)$.

We will impose the following conditions on the blind BV deblurring: the recorded image $u_0 \in L^2(\mathbb{R}^2)$, the original image $u \in \mathrm{BV}(\mathbb{R}^2)$, and the unknown PSF $k(x)$ is in $BV(\mathbb{R}^2)$ and satisfies

$$\int_{\mathbb{R}^2} k(x)dx = 1.$$

As $u \in \mathrm{BV}(\mathbb{R}^2)$, the Poincaré inequality (4.40) implies that $u \in L^2(\mathbb{R})$, hence

$$\|k \star u\|_{L^2} \le \|k\|_{L^1} \|u\|_{L^2}, \tag{4.42}$$

as follows from Young's inequality:

$$\|f \star g\|_{L^r} \le \|f\|_p \|g\|_q,$$

with

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q},$$

that we used with $r = 2$, $p = 1$ and $q = 2$. This may also be seen directly:

$$\begin{aligned}
\|k \star u\|_{L^2}^2 &= \int k(y)u(x-y)k(z)u(x-z)dxdydz \\
&= \int k(y)k(z) \left(\int u(x-z)u(x-y)dx \right) dydz \\
&\le \int k(y)k(z) \left(\int |u(x-y)|^2 dx \right)^{1/2} \left(\int |u(x-z)|^2 dx \right)^{1/2} dydz = \|k\|_{L^1}^2 \|u\|_{L^2}^2,
\end{aligned}$$

and (4.42) follows. The bound (4.42) means that under the above assumptions the functional $E(u, k; u_0)$ is well-defined.

The non-uniqueness of the minimizer

It is interesting that minimizer of the energy $E(u, k; u_0)$ need not be unique. An obvious source of the non-uniqueness comes from translation invariance: if (u_*, k_*) is a minimizer then the translates $u_a(x) = u(x - a)$, $k_a(x) = k(x - a)$ give a minimizer as well, for any $a \in \mathbb{R}^2$, since the energy of (u, k) is the same as that of (u_a, k_a).

Another reason for non-uniqueness is more special but also more amusing.

Theorem 4.4 *Suppose that $(u_*, k_*) \in BV(\mathbb{R}^2) \times BV(\mathbb{R}^2)$ is a minimizer of the energy*

$$E(u, k; u_0) = \alpha \int_\Omega |Du| + \frac{\lambda}{2} \int_{\mathbb{R}^2} (k \star u - u_0)^2 dx + \beta \int_{\mathbb{R}^2} |Dk|, \tag{4.43}$$

under the above assumptions on u_0 and k. Assume that

$$m = \int_{\mathbb{R}^2} u_*(x) dx = \frac{\beta}{\alpha}.$$

Then $(u_1, k_1) = (mk_*, k_*/m)$ must be a minimizer of $E(u, k; u_0)$ as well.

Proof. The kernel $k_1 = u_*/m$ satisfies the normalization

$$\int_{\mathbb{R}^2} k_1(x) dx = 1.$$

It is also immediate to see that $E(u_1, k_1; u_0) = E(u, k; u_0)$.

An even more dramatic non-uniqueness occurs for the minimization problem for a slightly different energy, with the BV-norm replaced by the Sobolev H^1-norm:

$$E_2(u, k; u_0) = \frac{\alpha}{2} \int_{\mathbb{R},^2} |\nabla u|^2 dx + \frac{\beta}{2} \int_{\mathbb{R}^2} |\nabla k|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} (k \star u - u_0)^2 dx. \tag{4.44}$$

This functional is minimized over $u \in H^1(\mathbb{R}^2)$ and $k \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with the condition

$$\int_{\mathbb{R}^2} k(x) dx = 1. \tag{4.45}$$

Let us define the Fourier transform as

$$\hat{f}(k) = \int_{\mathbb{R}^2} e^{-2\pi i k \cdot x} f(x) dx.$$

Then the normalization (4.45) is simply $\hat{k}(0) = 1$. Let us now choose a smooth real-valued odd function $\eta(\xi)$: $\eta(\xi) = -\eta(-\xi)$, and set

$$\hat{u}_1(\xi) = u_*(\xi) e^{i\eta(\xi)}, \quad \hat{k}_1(\xi) = u_*(\xi) e^{-i\eta(\xi)}.$$

Then the inverse Fourier transforms

$$u_1(x) = \int_{\mathbb{R}^2} e^{2\pi i \xi \cdot x} \hat{u}_1(\xi) d\xi, \quad k_1(x) = \int_{\mathbb{R}^2} e^{2\pi i \xi \cdot x} \hat{k}_1(\xi) d\xi$$

satisfy $\hat{u}_1(-\xi) = \hat{u}_1^*(\xi)$, $\hat{k}_1(-\xi) = \hat{k}_1^*(\xi)$ so that $u_1(x)$ and $k_1(x)$ are real, but more importantly we have

$$\hat{k}_1(\xi) \hat{u}_1(\xi) = \hat{k}(\xi) \hat{u}(\xi),$$

meaning that $k \star u(x) = k_1 \star u_1(x)$ for all $x \in \mathbb{R}^2$. Recall also that

$$\int |\nabla u(x)|^2 dx = (4\pi)^2 \int |\xi|^2 |\hat{u}(\xi)|^2 d\xi,$$

with a similar expression $k(x)$, meaning that

$$\int |\nabla u(x)|^2 dx = \int |\nabla u_1(x)|^2 dx, \quad \int |\nabla k(x)|^2 dx = \int |\nabla k_1(x)|^2 dx.$$

Therefore, we have

$$E_2(u, k; u_0) = E_2(u_1, k_1; u_0),$$

no matter which function $\eta(\xi)$ of this kind we take – this is another huge source of non-uniqueness.

Therefore, in order to get uniqueness of a minimizer we need to impose extra conditions on $k(x)$ to remove the above symmetries.

Existence of minimizers

In order to study the existence of minimizers of the energy

$$E(u, k; u_0) = \alpha \int_{\mathbb{R}^2} |Du| + \beta \int_{\mathbb{R}^2} |Dk| + \frac{\lambda}{2} \int_{\mathbb{R}^2} (k \star u - u_0)^2 dx, \tag{4.46}$$

we will use a slightly different space than $\mathrm{BV}(\mathbb{R}^2)$, namely

$$\mathrm{BV}_2(\mathbb{R}^2) = \left\{ u \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |Du| < +\infty \right\}.$$

The Poincaré inequality (4.40) implies that $\mathrm{BV}(\mathbb{R}^2) \subseteq \mathrm{BV}_2(\mathbb{R}^2)$, so that $\mathrm{BV}_2(\mathbb{R}^2)$ is, indeed, a bigger space.

As we have previously mentioned, we will require additional conditions to construct a minimizer, so we will assume the following: (i) the recording $u_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, (ii) the original image $u \in \mathrm{BV}_2(\mathbb{R}^2)$ and $\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty}$, and, finally, (iii) the PSF $k \in \mathrm{BV}(\mathbb{R}^2)$, $k(x) \geq 0$, and

$$\int_{\mathbb{R}^2} k(x) dx = 1.$$

Condition (ii) is quite natural: a "sufficiently rough" noise should increase the L^∞-norm, so it is reasonable to expect that $\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty}$.

In order to have additional compactness, we impose the following condition: (iv) there exists a nonnegative function $F \in L^1(\mathbb{R}^2)$ and some $R > 0$ so that

$$0 \leq k(x) \leq F(x), \quad \text{for all } x \in \mathbb{R}^2 \text{ with } |x| \geq R. \tag{4.47}$$

Here $F(x)$ and R are prescribed and are known a priori, that is, condition (4.47) is a constraint on the possible values of $k(x)$. For instance, we may require that $k(x)$ is compactly supported inside a ball B_R, and to enforce that we would take $F(x) \equiv 0$ for $|x| \geq R$. Let $\mathcal{A}$ be the set of all $(k, u) \in BV(\mathbb{R}^2) \times BV_2(\mathbb{R}^2)$ that satisfy the constraints (i)-(iv) above.

Theorem 4.5 *There exist minimizers $(u, k) \in BV_2(\mathbb{R}^2) \times BV(\mathbb{R}^2)$ for the energy functional (4.46) over the set $\mathcal{A}$.*

We will need the following stronger version of the Poincaré inequality in Theorem 4.3, dropping the assumption that $u \in L^1(\mathbb{R}^2)$.

Lemma 4.6 *(Gagliardo-Nirenberg inequality) For any $u \in BV_2(\mathbb{R}^2)$ we have*

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C \int_{\mathbb{R}^2} |Du|. \tag{4.48}$$

This is a particular case of the more general inequality: given $u \in L^{p^*}(\mathbb{R}^n)$ with

$$p^* = \frac{np}{n-p},$$

we have

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}. \tag{4.49}$$

The proof can be found in the book "Partial Differential Equations" by L.C. Evans, for instance.

We now prove Theorem 4.5. Let $R > 0$ be as in (4.47) and set

$$k_R(x) = \frac{1}{\pi R^2}\chi(|x| < R).$$

Then the kernel $k_R(x)$ has mass one:

$$\int_{\mathbb{R}} k_R(x)dx = 1,$$

it automatically satisfies the constraint (4.47) and

$$\int_{\mathbb{R}^2} |Dk| = \frac{2\pi R}{\pi R^2} = \frac{2}{R} < +\infty.$$

Therefore, $E(u \equiv 0, k_R; u_0) < +\infty$ and thus

$$\min_{(k,u)\in\mathcal{A}} E(u, k; u_0) < +\infty,$$

and a minimizing sequence $(u_n, k_n) \in \mathcal{A}$ exists. Our goal is to show that this sequence has to converge to a minimizer (u_*, k_*) of $E(u, k)$ over $\mathcal{A}$.

The Gagliardo-Nirenberg inequality (4.48) implies that u_n is a bounded sequence in $L^2(\mathbb{R}^2)$, hence for any bounded domain Ω the sequence u_n is bounded in $L^1(\Omega)$. Thus, as $TV(u_n) \leq (1/\alpha)E(u_n, k_n; u_0)$, the sequence u_n is uniformly bounded in $\mathrm{BV}(\Omega)$. Therefore, u_n has a convergent subsequence in $L^1(\Omega)$ (recall that Ω is a bounded domain). Then the standard diagonal argument implies that we can extract a subsequence u_n that converges in $L^1(B_R)$ to a function u_* for any ball B_R with $R > 0$ (to see that we first achieve this for integer radii $R \in \mathbb{N}$ which implies the claim for any $R > 0$). It is easy to see that the limit u_* does not depend on R. Extracting another subsequence we may ensure that

$$u_n(x) \to u(x) \text{ almost everywhere in } \mathbb{R}^2.$$

It follows that

$$\int_{\mathbb{R}^2} |Du_*| \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |Du_n|. \tag{4.50}$$

Similarly, we have the same result for the sequence k_n: after extracting a subsequence, it converges to a limit k_* in $L^1(B_R)$ for any ball B_R, and almost everywhere pointwise, so that

$$\int_{\mathbb{R}^2} |Dk_*| \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^2} |Dk_n|. \tag{4.51}$$

We will now show that (u_*, k_*) is a minimizer of $E(u,k)$ over the set $\mathcal{A}$.

Let us show that, after extracting a subsequence, the convolution also converges pointwise:

$$k_n \star u_n(x) \to k_* \star u_*(x), \text{ for any } x \in \mathbb{R}^2. \tag{4.52}$$

To this end, for any fixed $x \in \mathbb{R}^2$ we denote by $k^x(y) = k(x-y)$, and write

$$(k \star u)(x) = \langle k^x(y), u(y)\rangle = \langle k^x(y), u(y)\rangle_{B_r} + \langle k^x(y), u(y)\rangle_{B_r^c},$$

with $r = R + |x|$, where R is as in condition (4.47). The sequence k_n^x converges to k_*^x in $L^1(B_r)$, and, as we are minimizing the functional $E(u,k)$ over the functions u such that $|u(x)| \leq \|u_0\|_{L^\infty}$ for all $x \in \mathbb{R}^2$, we know that $|u_n(y)| \leq \|u_0\|_{L^\infty}$. Therefore, we have

$$|\langle k_n^x, u_n\rangle_{B_r} - \langle k_*^x, u_n\rangle_{B_r}| \leq \|u_0\|_{L^\infty} \|k_n^x - k_*^x\|_{L^1(B_r)} \to 0 \text{ as } n \to +\infty.$$

Moreover, we have, by the Lebesgue dominated convergence theorem,

$$\langle k_*^x, u_n\rangle_{B_r} \to \langle k_*^x, u_*\rangle_{B_r},$$

as u_n converges pointwise to u_*, the sequence u_n is bounded in L^∞, and $k_*^x \in L^1(B_r)$. Hence, we have, from the last two limits:

$$\langle k_n^x, u_n\rangle_{B_r} \to \langle k_*^x, u_*\rangle_{B_r}. \tag{4.53}$$

On the other hand, for the integral over B_r^c we may use property (4.47) and our choice of $r = R + |x|$. There, we have

$$|y - x| \geq |y| - |x| > r - |x| = R,$$

hence

$$|k_n^x(y)| = |k_n(x-y)| \leq F(x-y) = F^x(y), \text{ for any } y \in B_r^c, \text{ and all } n \geq 1. \tag{4.54}$$

Therefore, once again, as u_n is a sequence uniformly bounded in $L^\infty(\mathbb{R}^2)$, and because of (4.54), we can use the Lebesgue dominated convergence theorem, to show that

$$\langle k_n^x, u_n\rangle_{B_r^c} \to \langle k_*^x, u_*\rangle_{B_r^c}. \tag{4.55}$$

Together, (4.53) and (4.55) imply (4.52).

In order to finish the proof that (u_*, k_*) is a minimizer, we note that (4.52) implies (by Fatou's lemma) that

$$\int_{\mathbb{R}^2} (k_* \star u_* - u_0)^2 dx \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^2} (k_n \star u_n - u_0)^2 dx. \tag{4.56}$$

As we have already shown that the TV-norms of u_* and k_* satisfy (4.50) and (4.51), we conclude that (u_*, k_*) is a minimizer of $E(u, k)$ over $\mathcal{A}$.

It remains only to verify that u_* and k_* satisfy the conditions we have imposed on u and k: by construction, we have $\|u_*\|_{L^\infty} \leq \|u\|_{L^\infty}$, as u_* is a point-wise limit of u_n which all satisfy this condition. The Gagliardo-Nirenberg inequality and Fatou's lemma imply that

$$\|u_*\|_{L^2} \leq \liminf_{n \to \infty} \|u_n\|_{L^2} \leq \liminf_{n \to \infty} \int_{\mathbb{R}^2} |Du_n| < +\infty,$$

hence $u_* \in BV_2(\mathbb{R}^2)$. Similarly, we have $TV(k_*) < +\infty$, $k_* \geq 0$, and $k_* \in L^1(\mathbb{R}^2)$ by the Lebesgue dominated convergence theorem, whence $k \in BV(\mathbb{R}^2)$. Finally, the condition

$$\int_{\mathbb{R}^2} k_*(x) dx = 1,$$

holds since it is satisfied by each k_n, and by convergence of k_n to k_* in L^1_{loc} together with the upper bound (4.47).

5 Image inpainting

Image inpainting is a jargon for interpolation in the imaging context. There are, obviously, many difficult issues in this problem compared to some of the more mundane problems of function interpolation – one has to deal with "healing" very complex images.

5.1 Harmonic inpainting and its extensions

The simplest inpainting problem is to interpolate a smooth function $u(x)$ into a bounded domain D. The problem is to construct an approximation u_D to the restriction u^0_D of a smooth function $u^0(x)$ defined in a bigger domain Ω to the sub-domain $D \subseteq \Omega$. An inpainting scheme is said to be linear if

$$\|u_D - u^0_D\|_{L^\infty(D)} = O(d^2), \tag{5.1}$$

and, more generally it is of k-th order if

$$\|u_D - u^0_D\|_{L^\infty(D)} = O(d^{k+1}). \tag{5.2}$$

Here d is the diameter of D.

A simple way to design a linear scheme for smooth functions is via the harmonic extension. Let $f = u^0|_\Gamma$ be the restriction of u^0 to the boundary $\Gamma = \partial D$. Then the harmonic extension of f inside D is the solution of the boundary value problem

$$\begin{aligned} &\Delta u_h = 0 \text{ in } D, \\ &u_h = f \text{ on } \Gamma = \partial D. \end{aligned} \tag{5.3}$$

The anharmonic component $u^a = u^0 - u_h$ satisfies the Poisson equation

$$\begin{aligned} \Delta u_a &= \Delta u^0 \text{ in } D, \\ u_a &= 0 \text{ on } \Gamma = \partial D. \end{aligned} \tag{5.4}$$

While we do not know the forcing term Δu^0 in (5.4), we can still estimate (5.3) – we have the following approximation theorem.

Theorem 5.1 *Let d be the diameter of D, then we have*

$$\|u^a\|_{L^\infty(D)} \leq \frac{d^2}{4} \|\Delta u^0\|_{L^\infty(D)}. \tag{5.5}$$

Proof. Let $M = \|\Delta u^0\|_{L^\infty}$, then the maximum principle implies that $|u^a(x)| \leq M v(x)$, with the function $v(x)$ that solves

$$\begin{aligned} -\Delta v &= 1 \text{ in } D, \\ v &= 0 \text{ on } \Gamma = \partial D. \end{aligned} \tag{5.6}$$

It follows from the maximum principle that $v(x) > 0$ in D. Next, in order to bound $v(x)$ we take any point $z \in D$ and let $w(x)$ be the solution of the boundary value problem

$$\begin{aligned} -\Delta w &= 1 \text{ in } B(z;d), \\ w &= 0 \text{ on } \Gamma_1 = \partial B(z,d). \end{aligned} \tag{5.7}$$

The maximum principle implies that the function $w(x)$ is positive inside $B(z;d)$. Since d is the diameter of D, the ball $B(z,d)$ contains the set D, and, as $w(x) > 0 = v(x)$ on the boundary ∂D, we have $v(x) < w(x)$ in D (this is another consequence of the maximum principle). However, the function $w(x)$ is explicit:

$$w(x) = \frac{d^2 - |x-z|^2}{4}.$$

We conclude that $w(x) \leq d^2/4$ for all $x \in B(z;d)$, hence $v(x) \leq d^2/4$ for all $x \in D$, thus

$$|u^a(x)| \leq M \frac{d^2}{4}, \text{ for all } x \in D,$$

and we are done.

This theorem shows that harmonic extension gives a linear inpaiting. This can be further improved by, in a sense, iterating the harmonic extension. Let us write

$$u = u^h + u^a,$$

then, by construction, u^a satisfies the Poisson equation (5.4):

$$\begin{aligned} \Delta u^a &= \Delta u^0 \text{ in } D, \\ u^a &= 0 \text{ on } \Gamma = \partial D, \end{aligned} \tag{5.8}$$

with an unknown right hand side Δu^0. Let us consider a harmonic inpaiting w_h of Δu^0:

$$\begin{aligned} &\Delta w_h = 0 \text{ in } D, \\ &w_h = \Delta u^0 \text{ on } \Gamma = \partial D. \end{aligned} \tag{5.9}$$

This problem can be solved for w_h and the solution satisfies the estimate

$$\|w_h - \Delta u^0\|_{L^\infty(D)} \leq \frac{d^2}{4} \|\Delta^2 u^0\|_{L^\infty(D)}, \tag{5.10}$$

as in Theorem 5.1. We may then replace the unknown force Δu^0 in (5.8) by the known force w_h and obtain the following problem

$$\begin{aligned} &\Delta \tilde{u}^a = w_h \text{ in } D, \\ &\tilde{u}^a = 0 \text{ on } \Gamma = \partial D. \end{aligned} \tag{5.11}$$

The solution of this problem satisfies

$$\|\tilde{u}^a - u^a\|_{L^\infty(D)} \leq \frac{d^2}{4} \|w_h - \Delta u^0\|_{L^\infty(D)} \leq \frac{d^4}{16} \|\Delta^2 u^0\|_{L^\infty(D)}. \tag{5.12}$$

We conclude that $u^h + w_h$ gives a cubic approximation of u^0:

$$\|u^0 - u^h - w_h\|_{L^\infty(D)} \leq \frac{d^4}{16} \|\Delta^2 u^0\|_{L^\infty(D)}. \tag{5.13}$$

This procedure can be continued giving approximations of higher and higher order but the constants in the error estimates such as (5.13) will involve derivatives of u^0 of higher and higher order. Therefore, this iterative process will give "better and better" approximations only for very smooth functions u^0.

5.2 Inpainting via BV minimization

The TV inpainting model

The TV inpainting model is as follows: given an image u^0 outside of a domain D we define the inpainted image u as the minimizer of the functional

$$E(u) = \int_\Omega |\nabla u| dx + \frac{\lambda}{2} \int_{\Omega \setminus D} (u - u^0)^2 dx. \tag{5.14}$$

Here Ω is the overall domain, of which D is a "small" subset.

A common approach is to regularize this functional by considering

$$E_a(u) = \int_\Omega \sqrt{a^2 + |\nabla u|^2} dx + \frac{\lambda}{2} \int_{\Omega \setminus D} (u - u^0)^2 dx, \tag{5.15}$$

with a small $a > 0$. This model has the following interesting interpretation: let us take $\lambda = \infty$, which means that we believe that u^0 has no noise. This automatically implies that $u = u^0$ on $\Omega \setminus D$, and we need to minimize

$$\tilde{E}_a(u) = \int_D \sqrt{a^2 + |\nabla u|^2} dx, \tag{5.16}$$

subject to the constraint $u = u^0$ on $\Gamma = \partial D$. Consider the surface $z(x) = u(x_1, x_2)/a$, defined over D, then its surface is

$$S(z) = \int_D \sqrt{1 + |\nabla z|^2} dx = \frac{1}{a} \int_D \sqrt{a^2 + |\nabla u|^2} dx. \tag{5.17}$$

Therefore, the minimization problem for $\tilde{E}_a(u)$ can be restated as follows: minimize the area of the surface $z = u(x, y)$ given its boundary data $z(x_1, x_2) = u^0(x_1, x_2)/a$ on ∂D. This problem might not have a solution, so it may be relaxed to minimizing

$$\tilde{S}(z) = \int_D \sqrt{a^2 + |\nabla u|^2} dx + \frac{\mu}{2} \int_{\partial D} (z - u^0)^2 dl. \tag{5.18}$$

TV inpainting and an optical illusion

Here is a cute example from the Chan-Shen book of how the TV norm can create illusions. Let us assume that the domain Ω is a symmetric "cross" of two rectangles R_v and R_h so that R_v is wide and tall (but taller than wider), and R_h is wide in the horizontal directional but short in the vertical. We will need to inpaint the intersection $D = R_v \cap R_h = L \times l$, with $L > l$. Assume that the function u^0 is slightly darker in $R_h \setminus D$ than in $R_v \setminus D$:

$$u^0(x) = \frac{1}{2} - \varepsilon \text{ in } R_h \setminus D,$$

and

$$u^0(x) = \frac{1}{2} + \varepsilon \text{ in } R_h \setminus D.$$

The question is what will be the inpainted function $u(x)$ in D – we assume here that $\lambda = +\infty$. One can show that the minimizer is constant inside D: $u(x) \equiv c$ in D. The energy of such function is

$$E(c) = 2|c - 1/2 - \varepsilon|L + 2|c - 1/2 + \varepsilon|l = |2c - 1 - 2\varepsilon|L + |2c - 1 + 2\varepsilon|l.$$

This is a piece-wise linear function such that

$$E(1/2 - \varepsilon) = 4\varepsilon L,$$

and

$$E(1/2 + \varepsilon) = 4\varepsilon l.$$

hence, the minimum is achieved at $c = 1/2 + \varepsilon$, meaning that the whole interior of D will be lighter if $L > l$. That is, the inpainting picks the color of the wider rectangle.

Digital zooming as inpainting

Digital zooming is the problem of reconstructing a digital image u of size $2n \times 2m$ from a digital image u^0 of size $n \times m$. This problem is very similar to inpainting but is more appropriate in the discrete version than continuous. Let Ω be the fine grid to which we need to extend u^0 as a zoom-in. The original image u^0 is defined on a grid Ω_0 which is a sub-grid

of Ω. For each pixel $\alpha \in \Omega$ we define N_α as the set of all neighbors of the pixel α in Ω. At each pixel α we define

$$|\nabla u(\alpha)| = \sqrt{\sum_{\beta \in N_\alpha} (u_\beta - u_\alpha)^2}.$$

We also define

$$\lambda_e(\alpha) = \begin{cases} \lambda, & \alpha \in \Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

The digital zoom-in problem is to minimize the discrete energy

$$E(u) = \sum_{\alpha \in \Omega} |\nabla_\alpha u| + \sum_{\alpha \in \Omega} \frac{\lambda_e(\alpha)}{2} (u_\alpha - u_\alpha^0)^2,$$

over all digital images u defined on the fine grid Ω.