Homework #6

1. (i) Let $f$ and $h$ be two functions of the Schwartz class, and 

$$g(x) = f * h(x) = \int f(x - y)h(y)dy.$$ 

Show that the Fourier transform of $\hat{g}(\xi) = \hat{f}(\xi)\hat{h}(\xi)$.

(ii) Let $K(x)$ be a rapidly decaying smooth function. Consider the equation for $t > 0$ and $x \in \mathbb{R}$:

$$\frac{\partial u(t, x)}{\partial t} = \int_{-\infty}^{\infty} K(x - y)u(t, y)dy,$$

with the initial data $u(t = 0, x) = f(x)$. Show that if $f(x)$ is a Schwartz class function then $u(t, x)$ is smooth and bounded, as a function of $x$, for each $t > 0$ fixed. What can you say about 

$$\int |u(t, x)|^2 dx$$

if, in addition, you know that $\hat{K}(\xi) \leq 0$ for all $\xi \in \mathbb{R}$?

2. (i) Let $\phi(t, x)$ solve the wave equation

$$\frac{1}{c^2} \phi_{tt} - \phi_{xx} = 0.$$ 

Show that $u = \phi_t$ and $p = -\phi_x$ satisfy the $2 \times 2$ system

$$\frac{1}{c^2} u_t + p_x = 0$$

$$p_t + u_x = 0.$$ 

(ii) Let $A$ be a symmetric positive definite $n \times n$ matrix, $D$ a symmetric matrix, and let $v = (v_1, \ldots, v_n)$ be the vector solution of the system

$$Av_t + Dv_x = 0,$$

with the initial data $v(0, x) = v_0(x)$ that is smooth and rapidly decaying at infinity. Use the Fourier transform to find the solution of the initial value problem. Hint: keep in mind that solving ODE systems of the form $y'(t) = By$ is easier after $B$ is diagonalized.

(iii) Put the $2 \times 2$ system in part (i) into the form as in in part (ii), apply the result of (ii), and recover the d’Alembert formula.

(iv) Show that in the setting of (ii) the energy defined as

$$E(t) = \int \langle Av(t, x), v(t, x) \rangle dx$$

is a conserved quantity: $E(t) = E(0)$ for all $t \geq 0$. What is the corresponding energy for the wave equation?

(v) Generalize this to two dimensions: let $v(t, x, y) = (v_1, \ldots, v_n)$ solve the $n \times n$ system

$$v_t + D_1v_x + D_2v_y = 0,$$
with the initial data \( v(0, x) = v_0(x) \) that is smooth and rapidly decaying at infinity. Here \( D_1 \) and \( D_2 \) are symmetric matrices. Show that the energy

\[
E(t) = \int |v(t, x, y)|^2 dx dy
\]

is conserved. Assume that for any \( k = (k_1, k_2) \) all eigenvalues of the matrix \( D(k_1, k_2) = k_1D_1 + k_2D_2 \) are simple. Solve the initial value problem using the Fourier transform and explain the role of the eigenvalues and eigenvectors of the matrix \( D(k) \).

3. Compute explicitly the entropy solution for the Burgers’ equation

\[
u_t + uu_x = 0,
\]

with the initial data

\[
g(x) = \begin{cases} 
1 & \text{if } x < 0, \\
2 & \text{if } 0 < x < 1, \\
0 & \text{if } 1 < x,
\end{cases}
\]

and explain what happens at various times \( t > 0 \). Sketch the graph of \( u(t, x) \) at various times when it behaves differently.

4. Consider the following two regularizations of the Burgers’ equation:

\[
u_t + uu_x = \varepsilon u_x, \\
v_t + vv_x = \varepsilon v_{xx},
\]

where \( \varepsilon > 0 \) is a fixed number (you should think of it as being small).

(i) Look for solutions of the first equation of the form \( u(t, x) = U(x - ct) \) with \( U(x) \) such that \( U(x) \to 1 \) as \( x \to -\infty \), \( U(x) \to 0 \) as \( x \to +\infty \) and appropriately chosen \( c \), and such that \( 0 < U(x) < 1 \). (Hint: get an ODE for \( U(x) \) and solve it.) You may assume that all derivatives of \( U(x) \) tend to zero at infinity. This is a viscous approximation of the inviscid shock.

(ii) Show that the second equation has no such solution for \( v(t, x) \).

5. (i) Show that if \( v(t, x) \) satisfies

\[
\varepsilon v_t = v_{xx} + v_x^2,
\]

then \( u = e^v \) is the solution of the heat equation

\[
u_t = u_{xx}.
\]

(ii) Find an analogous transformation for the solution of

\[
v_t = \varepsilon v_{xx} + v_x^2.
\]

Here, we think of \( \varepsilon > 0 \) as a small parameter. Use this idea to find the solution \( w(t, x) \) for a given initial condition \( w(0, x) = f(x) \). Let \( \varepsilon \to 0 \) and show that if \( f(x) \) is smooth then the limit \( \tilde{w}(t, x) = \lim_{\varepsilon \to 0} w(t, x) \) is continuous. Show that the derivative \( \tilde{w}_x(t, x) \) may not exist at some \( x \) for \( t > 0 \) even if \( f(x) \) is smooth.