Homework # 3.

Problem 1. Let $u(t, x)$ satisfy the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \quad t > 0, \quad x \in \mathbb{R}^3$$

in three dimensions, with the initial condition $u(0, x) = f(x)$, where $f(x)$ is a continuous function that vanishes outside of the ball $\{|x| \leq 1\}$. Define

$$v(x) = \int_0^\infty u(t, x) dt.$$

(i) Use the explicit expression for $u(t, x)$ in terms of the heat kernel to show that this function is well-defined: the integral above converges.

(ii) Show that $v(x)$ is the unique bounded solution of the Poisson equation

$$-\Delta v = f(x), \quad x \in \mathbb{R}^3.$$

This gives a way to express solutions of the Poisson equation in terms of the solution of the initial value problem for the heat equation.

(iii) Use this to obtain the expression

$$H(x) = \frac{1}{4\pi|x|}$$

for the fundamental solution of the Laplace equation, starting with the heat kernel in three dimensions

$$G(t, x) = \frac{1}{(4\pi t)^{3/2}} e^{-|x|^2/(4t)}.$$

Problem 2. Let the function $g(t)$ be bounded and continuous, with $g(0) = 0$, and set

$$u(t, x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-x^2/(4(t-s))} g(s) ds,$$

defined for $t > 0$ and $x > 0$.

(i) Why does the integral above converge for all $x > 0$ (there is a potential problem at $t = s$ where $1/(t-s)^{3/2}$ blows up)?

(ii) Verify that $u(t, x)$ satisfies the heat equation for $t > 0$ and $x > 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

(iii) Show that $u(t, x) \to 0$ as $t \to 0$ for all $x > 0$.

(iv) Show that $u(t, x) \to g(t)$ as $x \to 0$ for all $t > 0$. Hence, $u(t, x)$ is the solution of the initial boundary value problem posed in the half-space $x > 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

$$u(t, 0) = g(t),$$

$$u(0, x) = 0.$$
Problem 3. Write down an explicit formula for the solution of the initial value problem
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} + cu = 0, \quad t > 0, \quad x \in \mathbb{R},
\]
\[u(0, x) = f(x),\]
where \(b, c \in \mathbb{R}\) are fixed constants. Hint: use a change of variables to get rid of the term with the first derivative. Show that if \(f(x) \geq 0\) for all \(x \in \mathbb{R}^n\), then \(u(t, x) \geq 0\) for all \(t > 0\) and \(x \in \mathbb{R}^n\).

Problem 4. Let \(F(x)\) be a strictly convex function defined for \(x \in \mathbb{R}^n\) such that for any \(M > 0\) there exists \(R > 0\) so that \(F(x) > M\) if \(|x| > R\).

(i) Show that \(F\) has a unique minimum \(x_0\).
(ii) Consider an ordinary differential equation
\[
\frac{dX}{dt} = -\nabla F(X(t)), \quad X(0) = x_1,
\]
with some \(x_1 \in \mathbb{R}^n\). Show that \(G(t) = F(X(t))\) is a decreasing function of \(t\).
(iii) Show that \(X(t)\) converges to \(x_0\) as \(t \to +\infty\). This is the gradient descent method.

Problem 5. Consider the functional
\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx,
\]
defined for \(u \in A = \{u \in C^2(\Omega)\}\) such that \(u = 0\) on \(\partial \Omega\). Fix one such \(u\), and consider \(G(s; u, v) = I(u + sv)\), for various \(v \in A\).

(i) Find a function \(g\), so that for any \(v \in A\) we have
\[
\frac{dG}{ds}(s = 0; u, v) = \int_{\Omega} g(x)v(x) dx.
\]
(ii) Relate the heat equation
\[
\begin{align*}
  u_t &= \Delta u, \quad \text{in} \ \Omega, \\
  u &= 0, \quad \text{on} \ \partial \Omega, \\
  u(0, x) &= f(x), \quad \text{in} \ \Omega
\end{align*}
\]
to the gradient descent method for the functional \(I(u)\) starting at \(f\).

Problem 6. Consider the heat equation on the line
\[
\frac{\partial u}{\partial t} = u_{xx},
\]
with the initial condition \(u(0, x) = f(x)\), with \(f(x) = 1\) if \(-1 \leq x \leq 1\) and \(f(x) = 0\) otherwise.

(i) Show that the solution \(u(t, x)\) is even in \(x\) for all \(t > 0\): \(u(t, x) = u(t, -x)\) and \(\frac{\partial u(t, x)}{\partial x} < 0\) for all \(t > 0\) and \(x > 0\).
(ii) Write a numerical algorithm such that it does approximate the solution of the above problem well (whatever it means), but the numerical solution is
not monotonically decreasing in $x$ for $x > 0$ but has some small oscillations. In other words, your algorithm should be bad in a special way.