Homework # 4.

Problem 1. Let \( a(x) \) be a smooth function such that \( 1 < a(x) < 10 \) for all \( x \in \mathbb{R} \), and let \( b \in \mathbb{R} \). Let \( u(t, x) \) be a bounded solution (you may assume it exists and is unique) of the initial value problem
\[
\frac{\partial u}{\partial t} - a(x) \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x \in \mathbb{R}^n
\]
\[u(0, x) = f(x).\]
Assume that the function \( f(x) \) is smooth and compactly supported. Show that \( u(t, x) \) attains its maximum over all \( t \geq 0 \) and \( x \in \mathbb{R} \) at the time \( t = 0 \), that is,
\[u(t, x) \leq \max_{y \in \mathbb{R}} f(y),\]
for all \( t > 0 \) and \( x \in \mathbb{R} \).

Problem 2. Let \( u(t, x) \) satisfy the heat equation
\[
\frac{\partial u}{\partial t} - \Delta u = 0, \quad t > 0, \quad x \in \mathbb{R}.
\]
in one dimension, with the initial condition \( u(0, x) = f(x) \).

(i) Use the Fourier transform to show that
\[u(t, x) = \int e^{2\pi ikx - 4\pi^2 k^2 t} \hat{f}(k) dk,
\]
where \( \hat{f}(k) \) is the Fourier transform of the function \( f \).

(ii) Show that the above representation is the same as
\[u(t, x) = \int G(t, x - y) f(y) dy,
\]
where \( G(t, x) \) is the one-dimensional heat kernel \( G(t, x) = (1/\sqrt{4\pi t}) e^{-|x|^2/4t} \).

Problem 3. Consider the function \( f(x) \) defined as
\[f(x) = \frac{1}{2} - x, \quad \text{for } 0 < x < 1,
\]
and extended periodically to all of \( \mathbb{R} \) (the resulting function has a jump discontinuity at all integers \( x \in \mathbb{Z} \)).

(i) Show that \( \hat{f}_n = 1/(2\pi i n) \) for \( n \neq 0 \) and \( \hat{f}_0 = 0 \).

(ii) Show that
\[(S_N f)'(x) = D_N(x) - 1,
\]
where \( D_N(x) = \sin((2N + 1)\pi x)/\sin(\pi x) \) is the Dini kernel, and that
\[S_N f(x) = -x + \int_0^x \frac{\sin((2N + 1)\pi t)}{\sin(\pi t)} dt.
\]

(iii) Let
\[\tilde{S}_N f(x) = -x + \int_1^x \frac{\sin((2N + 1)\pi t)}{\pi t} dt.
\]
and show that there exists some constant $C > 0$ so that $|S_N f(x) - \tilde{S}_N f(x)| \leq C/N$ for all $x \in [0, 1/4]$.

(iv) Let $x_N = 1/(2N)$ and show that
\[
\tilde{S}_N f(x_N) \to \tilde{S} = \frac{1}{\pi} \int_0^\pi \frac{\sin t}{t} dt.
\]

(v) Show that $\tilde{S} > 1/2$ (use a computer if you need) and explain why anyone would care (even if you don’t).

Problem 4. (i) Let $a_k$ be the Fourier coefficients of the function $u(x) = 1/2 - x$, $0 < x < 1$, extended periodically:
\[
a_k = \int_0^1 \left(\frac{1}{2} - x\right) e^{-2\pi ikx} dx.
\]
Show that there exists a constant $M > 0$ so that for all $x \in [0, 1]$ and all $N \in \mathbb{Z}$ the partial sums of the Fourier series satisfy
\[
\left| \sum_{k=-N}^{N} a_k e^{2\pi ikx} \right| \leq M.
\]

(ii) Let two integer sequences $N_k \geq 1$ and $m_k \geq 1$ be such that
\[
N_{k+1} - m_{k+1} > N_k + m_k,
\]
and define
\[
f(x) = \sum_{k=1}^{\infty} e^{2\pi i N_k x} \frac{B_k(x)}{k^2},
\]
with
\[
B_k(x) = \sum_{j=-m_k}^{m_k} a_j e^{2\pi ijx}.
\]
Show that the function $f(x)$ is continuous and find its Fourier coefficients in terms of $a_k$.

(iii) Show that the partial Fourier sums $S_{N_k} f(0)$ satisfy a lower bound
\[
S_{N_k} f(0) \geq \frac{C \log m_k}{k^2}.
\]

(iv) Find a choice of $m_k$ and $N_k$ so that the Fourier series of $f(x)$ at $x = 0$ diverges.

Problem 5. Consider an equation of the form
\[
u_{xx} + u(1 - u^2) = 0,
\]
posed on the real line $-\infty < x < +\infty$. Find a steady solution of the form $u(t, x) = v(x)$ such that $v(-\infty) = -1$, $v(+\infty) = 1$, and $-1 < v(x) < 1$ for all $x \in \mathbb{R}$. Next, ask the same question for a similar problem
\[
\varepsilon u_{xx} + u(1 - u^2) = 0,
\]
now with $\varepsilon > 0$ but small. What will happen as $\varepsilon \to 0$?