Problem 1.

Define \( v(t,x) = u(t,x + bt) \), then

\[
\frac{\partial v}{\partial t} (t,x) - a(x+bt) \frac{\partial^2 u}{\partial x^2} (t,x) = 0, \forall t > 0, x \in \mathbb{R},
\]

\( v(0,x) = f(x) \).

It suffices to show that \( v(t,x) \leq F = \max_{y \in \mathbb{R}} f(y) \).

We consider the domain \( U_{T,L} = \{(t,x) : 0 \leq t \leq T, -L \leq x \leq L\} \) where \( T \) and \( L \) are some positive number and the auxiliary function

\[
w(t,x) = v(t,x) - F - M L^2 (\bar{a} - a(x + bt)),
\]

where \( \bar{a} = 10 > a(x) \), \( F = \sup_{y \in \mathbb{R}} f(y) \), and \( M = |\sup_{t \geq 0, x \in \mathbb{R}} v(t,x)| + |F| \) (since \( u(t,x) \) is a bounded solution, \( M \) is a well defined real number).

We have

\[
\frac{\partial w}{\partial t} (t,x) - a(x+bt) \frac{\partial^2 w}{\partial x^2} (t,x) = -\frac{2M}{L^2} (\bar{a} - a(x + bt)) < 0, \forall (t,x) \in U_{T,L},
\]

\( w(0,x) \leq 0, \forall x \in (-L, L) \),

\( w(t, \pm L) \leq 0, \forall t \in [0, T] \).

By a similar reasoning as in Theorem 1.16 in the notes, we can show that \( w(t,x) \) must attain its maximum over \( U_{T,L} \) on the parabolic boundary \( \Gamma_{T,L} = \{(t,x) : t = 0 \land -L < x < L \lor 0 \leq t \leq T \land x = \pm L\} \).

The reasoning:

Suppose \( w(t,x) \) attains its maximum at \( (t_0, x_0) \in U_{T,L} \setminus \Gamma_{T,L} \), then

\[
\frac{\partial w}{\partial t} (t_0, x_0) \geq 0, \frac{\partial^2 w}{\partial x^2} (t_0, x_0) \leq 0,
\]

\[
\frac{\partial w}{\partial t} (t_0, x_0) - a(x_0 + bt_0) \frac{\partial^2 w}{\partial x^2} (t_0, x_0) \geq 0,
\]

which contradicts \( \frac{\partial w}{\partial t} (t_0, x_0) - a(x_0 + bt_0) \frac{\partial^2 w}{\partial x^2} (t_0, x_0) < 0 \). Thus \( w \) must attain its maximum on \( \Gamma_{T,L} \).

Thus we have that \( w(t,x) \leq 0, \forall x \in U_{T,L} \), i.e.,

\[
v(t,x) \leq F + \frac{M}{L^2} (2\bar{a}t + x^2), \forall t \in [0,T], x \in [-L,L].
\]
Now, for any point \((t_1, x_1) \in \mathbb{R}^+ \times \mathbb{R}\), we want to show that \(v(t_1, x_1) \leq F\). First we set \(T = t_1\), and then we choose \(L > |x_1|\). Then we have
\[
v(t_1, x_1) \leq F + \frac{M}{L^2}(2at_1 + x_1^2).
\]
Now, note that the inequality holds for any \(L > |x_1|\). Let \(L \to \infty\) and we have
\[
v(t_1, x_1) \leq F.
\]

**Problem 2.**

(i) By Fourier transform we have
\[
\frac{\partial \hat{u}}{\partial t}(t, k) + 4\pi^2 k^2 \hat{u}(t, k) = 0, \forall t > 0, k \in \mathbb{R},
\]
\[
\hat{u}(0, k) = \hat{f}(k).
\]
Solve the equation we have
\[
\hat{u}(t, k) = e^{-4\pi^2 k^2 t}\hat{f}(k),
\]
\[
u(t, x) = \int \hat{u}(t, k)e^{2\pi ikx} dk = \int e^{2\pi i kx - 4\pi^2 k^2 t}\hat{f}(k) dk.
\]

(ii)
\[
u(t, x) = \int e^{2\pi i kx - 4\pi^2 k^2 t}\hat{f}(k) dk
\]
\[
= \int e^{2\pi i kx - 4\pi^2 k^2 t}\int e^{-2\pi iky} f(y) dy dk
\]
\[
= \int \left( \int e^{2\pi i k(x-y) - 4\pi^2 k^2 t} dk \right) f(y) dy
\]
\[
= \frac{1}{\sqrt{4\pi t}} \int \left( \int e^{-\pi \xi^2} e^{2\pi i \xi((x-y)/\sqrt{4\pi t})} d\xi \right) f(y) dy \quad (\xi = \sqrt{4\pi t} k)
\]
\[
= \frac{1}{\sqrt{4\pi t}} \int e^{-\pi ((x-y)/\sqrt{4\pi t})^2} f(y) dy
\]
\[
= \int G(t, x-y)f(y) dy
\]

**Problem 3.**

(i)
\[
\hat{f}_0 = \int_0^1 \frac{1}{2} - x \, dx = 0.
\]
\[
\hat{f}_n = \int_0^1 e^{-2\pi inx} \left( \frac{1}{2} - x \right) \, dx = \left( -\frac{e^{-2\pi inx}}{4\pi in} + \frac{x e^{-2\pi inx}}{2\pi in} - \frac{e^{-2\pi inx}}{4\pi n^2} \right) \bigg|_{x=0}^{x=1} = \frac{1}{2\pi in} \forall n \neq 0.
\]
(ii)

\[(S_N f)'(x) = \left( \sum_{k=-N}^{N} \hat{f}_k e^{2\pi i k x} \right)' = \sum_{k=-N}^{N} (2\pi i k \hat{f}_k) e^{2\pi i k x}
\]

\[= \sum_{k=-N}^{N} e^{2\pi i k x} - 1
\]

\[= D_N(x) - 1,
\]

and \(S_N f(0) = \sum_{k=-N}^{N} \hat{f}_k = 0\), thus

\[S_N f(x) = \int_{0}^{x} (D_N(t) - 1) \, dt = \int_{0}^{x} \frac{\sin((2N + 1)\pi t)}{\sin(\pi t)} \, dt - x.
\]

(iii)

\[S_N f(x) - \tilde{S}_N f(x) = \int_{0}^{x} \left( \frac{1}{\sin(\pi t)} - \frac{1}{\pi t} \right) \sin((2N + 1)\pi t) \, dt.
\]

Define \(g(z) = \frac{1}{\sin(z)} - \frac{1}{z}\), we have

\[g(z) = \frac{1}{\sin(z)} - \frac{1}{z} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots - \frac{1}{z}
\]

\[= \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \ldots
\]

\[= \frac{z}{3!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots
\]

\[= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \ldots
\]

so \(z = 0\) is a removable singular point. We define \(g(0) = 0\), then \(g(z)\) is a \(C^\infty\) function on \([0, \pi/4]\), then

\[h(t) := \frac{1}{\sin(\pi t)} - \frac{1}{\pi t} \in C^\infty[0, 1/4],
\]

then

\[
\forall x \in [0, 1/4],
\]

\[
\left| \int_{0}^{x} \left( \frac{1}{\sin(\pi t)} - \frac{1}{\pi t} \right) \sin((2N + 1)\pi t) \, dt \right|
\]

\[
= \left| \int_{0}^{x} h(t) \sin((2N + 1)\pi t) \, dt \right|
\]

\[
= \left| - h(t) \cos((2N + 1)\pi t) \bigg|_{t=0}^{t=x} + \int_{0}^{x} h'(t) \cos((2N + 1)\pi t) \, dt \right|
\]

\[
\leq \frac{C_1 + C_2}{(2N + 1)\pi},
\]

3
where
\[ C_1 = 2 \sup_{t \in [0, 1/4]} |h(t)|, \quad C_2 = \int_0^{1/4} |h'(t)| \, dt. \]

Define \( C = C_1 + C_2 \), then
\[ \left| \int_0^x h(t) \sin((2N + 1)\pi t) \, dt \right| \leq C/N, \forall x \in [0, 1/4]. \]

(iv)
\[
\tilde{S}_N f(x_N) = -1/(2N) + \int_0^{1/(2N)} \frac{\sin((2N + 1)\pi t)}{\pi t} \, dt \\
= -1/(2N) + \frac{1}{\pi} \int_0^{(2N+1)\pi/(2N)} \frac{\sin(y)}{y} \, dy \ (y = (2N + 1)/2N) \\
\Rightarrow \lim_{N \to \infty} \tilde{S}_N f(x_N) \\
= \lim_{N \to \infty} \left( -1/(2N) + \frac{1}{\pi} \int_0^{(2N+1)\pi/(2N)} \frac{\sin(y)}{y} \, dy \right) \\
= \frac{1}{\pi} \int_0^\pi \frac{\sin(y)}{y} \, dy.
\]

(v) Consider the function
\[
r(x) := \frac{\pi - x}{\pi}, \quad p(x) := \frac{\sin(x)}{x}.
\]

What we want to prove is that
\[
\frac{1}{\pi} \int_0^\pi p(x) \, dx > 1/2.
\]

Obviously, we have
\[
\frac{1}{\pi} \int_0^\pi r(x) \, dx = 1/2
\]
we just need to prove \( p(x) \neq r(x) \) (which is obvious) and \( p(x) \geq r(x) \ (\forall x \in [0, \pi]) \):
First, we have
\[
p(x) - r(x) = \frac{\sin(x)}{x} - \frac{\pi - x}{\pi} = \frac{\pi \sin(x) - x(\pi - x)}{\pi x}.
\]
Now we consider the function \( q(x) := \pi \sin(x) - x(\pi - x) \). We need to show that \( q(x) \geq 0 \), \( \forall x \in [0, \pi] \). Because of the symmetry of \( q(x) \) (which means that \( q(x) = q(\pi - x), \forall x \in [0, \pi] \)), we
just need to show \( q(x) \geq 0, \forall x \in [0, \pi/2] \). Note that \( q(0) = 0, q'(x) = \pi \cos(x) - \pi + 2x \),
\( q'(x) \) is concave on \([0, \pi/2]\), and \( q'(0) = q'(\pi/2) = 0 \), we have
\[
q'(x) \geq 0, \forall x \in [0, \pi/2],
\]
\[
q(x) \geq 0, \forall x \in [0, \pi/2],
\]
\[
p(x) \geq r(x), \forall x \in [0, \pi],
\]
\[
\bar{S} > 1/2.
\]
This means, as \( N \) goes to infinity, there exists \( x_N = 1/(2N) \), which goes to 0, such that
\( S_N f(x_N) \) goes to some constant which is strictly greater than 1/2. But we want to use
\( S_N f(x) \) to approximate \( f(x) \), and we know that \( |f(x)| \leq 1/2 \). So the sequence \( S_N f(x) \) does
not converge to \( f(x) \) uniformly. This is mainly because \( f(x) \) is not continuous at \( x = 0 \) (after
we extend \( f \) periodically). This is the Gibbs phenomenon. If we want to use \( S_N f(x) \) as
an approximation of \( f(x) \), then near \( x = 0 \), we need \( N \) to be large to get a relatively good
approximation.

**Problem 4.**

(i) From Problem 3.(ii) we know
\[
S_N f(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx} = -x + \int_{0}^{x} \frac{\sin((2N + 1)\pi t)}{\sin(\pi t)} \, dt.
\]
It’s obvious that the first term \(-x\) is bounded for \( x \in [0, 1] \). We only need to show that the
second term is bounded.

We note that the function \( h(t) := \frac{\sin((2N + 1)\pi t)}{\sin(\pi t)} \) is even w.r.t. \( x = 1/2 \), i.e., \( h(x) = h(1-x) \)
for any \( x \in [0, 1] \), thus it suffices to show that
\[
\left| \int_{0}^{x} \frac{\sin((2N + 1)\pi t)}{\sin(\pi t)} \, dt \right| \leq M, \quad \forall x \in [0, 1/2].
\]
To show this, we notice that the denominator of \( h(t) \) is a positive increasing function on
\((0, 1/2]\) and the numerator of \( h(t) \) is a periodic function which oscillates with period \( O(1/N) \).
If we split the interval \([0, x]\) into smaller chunks:
\[
[0, 1/(2N + 1)], \quad [1/(2N + 1), 2/(2N + 1)], \quad \ldots, \quad [(k/2N + 1), x],
\]
and we compute the integral on each chunk as
\[
b_j = \int_{j/(2N+1)}^{(j+1)/(2N+1)} h(t) \, dt,
\]
then the integral over \([0, x]\) can be written as
\[
\int_{0}^{x} \frac{\sin((2N + 1)\pi t)}{\sin(\pi t)} \, dt = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{k-1}b_k + (-1)^{k}b_{k+1},
\]
where \( b_j \) is a non-negative sequence and \( b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_k \geq b_{k+1} \).

The point of doing this is to rewrite the integral as an alternating sum of the sequence \( \{b_j\} \). Then we see that the integral can be bounded by \( b_1 \) due to the alternating property. We also have that

\[
b_1 = \int_0^{1/(2N+1)} h(t) \, dt \leq 1.
\]

Thus the integral is bounded.

(ii) We define

\[
g_k(x) := e^{2\pi i N_k x} \frac{B_k(x)}{k^2},
\]

then from (i) we know that \( |g_k(x)| \leq M/k^2 \).

Since each \( g_k(x) \) is continuous and the dominating series \( \sum_{k=1}^{\infty} M/k^2 \) has finite sum, we know that \( f(x) \) is continuous.

For the Fourier coefficients, we expand the expression for \( f(x) \) as

\[
f(x) = \sum_{k=1}^{\infty} \sum_{j=-m_k}^{m_k} \frac{a_j}{k^2} e^{2\pi i (N_k+j)x}.
\]

Since \( N_k + m_k < N_{k+1} - m_{k+1} \), we see that the terms for different \( k \) in the above expansion do not share the same exponentials. If we write the Fourier expansion of \( f(x) \) as \( f(x) = \sum_{l=-\infty}^{\infty} c_l e^{2\pi i l x} \), then by comparing the coefficients we have

\[
c_l = \begin{cases} 
\frac{a_j}{k^2}, & \text{if } l \text{ can be written as } l = N_k + j \text{ for some } k \in \mathbb{N}^+ \text{ and } -m_k \leq j \leq m_k, \\
0, & \text{otherwise.}
\end{cases}
\]

(iii) For \( S_{N_k} f(x) \), we only need to keep the terms in the Fourier expansion where \(-N_k \leq l \leq N_k \).

That gives

\[
S_{N_k} f(x) = \sum_{p=1}^{k-1} e^{2\pi i N_p x} \frac{B_p(x)}{p^2} + e^{2\pi i N_k x} \frac{0}{k^2} \sum_{j=-m_k}^{0} a_j e^{2\pi i j x}.
\]

Note that \( B_p(0) = 0 \) and we have

\[
S_{N_k} f(0) = \frac{1}{k^2} \sum_{j=-m_k}^{0} a_j = \frac{1}{k^2} \sum_{j=-m_k}^{-1} \frac{1}{2\pi ij} = \frac{i}{k^2} \sum_{j=1}^{m_k} \frac{1}{j}.
\]

Now the harmonic series \( \sum_{j=1}^{m_k} 1/j \) is bounded below by \( C \log(m_k) \) for some \( C > 0 \), thus

\[
|S_{N_k} f(0)| \geq \frac{C \log(m_k)}{k^2}.
\]
(iv) We set $m_k = \lceil e^{k^2} \rceil$ and we define $N_k$ recursively by
\[ N_1 = m_1 + 1, \]
\[ N_{k+1} = m_k + N_k + m_{k+1} + 1, \quad k = 1, 2, \ldots, \]
Then $N_{k+1} - mk + 1 > N_k + m_k$ is satisfied and $\log(m_k)/k^2 \geq k$, whose sum $\sum_{k=1}^{\infty} k$ diverges to $\infty$, which means the Fourier series of $f(x)$ at $x = 0$ diverges.

**Problem 5.**

We have $v_{xx} + v(1 - v^2) = 0$. Multiply by $v_x$ on both sides we have
\[ v_x v_{xx} + v(1 - v^2)v_x = 0, \]
\[ (v_x^2/2)_x + (v^2/2)_x - (v^4/4)_x = 0, \]
\[ v_x^2 + v^2 - v^4/2 = C_1, \]
\[ v_x^2 = (1 - v^2)^2/2 + C_2, \]
where $C_2 = C_1 - 1/2$.

Note that $v(-\infty) = -1, v(+\infty) = 1$ and $-1 < v(x) < 1$, we have
\[ \lim_{x \to \infty} v_x(x)^2 = C_2. \]

But $v(+\infty) = 1$ exits, so $C_2 = 0$, and
\[ v_x = \pm(1 - v^2)/\sqrt{2}. \]

Since $v(+\infty) > v(-\infty)$, we should choose the branch
\[ v_x = (1 - v^2)/\sqrt{2}. \]

Then we have
\[ \frac{dv}{1 - v^2} = dx/\sqrt{2}, \]
\[ \log \frac{1 + v}{1 - v} = \sqrt{2}(x - C_3), \]
\[ v(x) = \frac{e^{\sqrt{2}(x - C_3)} - 1}{e^{\sqrt{2}(x - C_3)} + 1} = \tanh \left( \frac{x - C_3}{\sqrt{2}} \right), \]
where $C_3$ is an arbitrary number.

If the equation is
\[ \varepsilon v_{xx} + v(1 - v^2) = 0, \]
then by a similar calculation we have
\[ v(x) = \tanh \left( \frac{x - C_3}{\sqrt{2\varepsilon}} \right). \]
As \( \varepsilon \to 0 \), we have

\[
v(x) \to \text{sgn}(x - C_3) = \begin{cases} 
-1, & \forall x < C_3, \\
0, & \text{if } x = C_3, \\
1, & \forall x > C_3.
\end{cases}
\]

This means, as \( \varepsilon \to 0 \), we have a jump at the point \( x = C_3 \).