Homework # 3.

Problem 1. Let \( u(t, x) \) satisfy the heat equation
\[
\frac{\partial u}{\partial t} - \Delta u = 0, \ t > 0, \ x \in \mathbb{R}^3
\]
in three dimensions, with the initial condition \( u(0, x) = f(x) \), where \( f(x) \) is a continuous function that vanishes outside of the ball \( \{|x| \leq 1\} \). Define
\[
v(x) = \int_0^\infty u(t, x)dt.
\]
(i) Use the explicit expression for \( u(t, x) \) in terms of the heat kernel to show that this function is well-defined: the integral above converges.
(ii) Show that \( v(x) \) is the unique bounded solution of the Poisson equation
\[
-\Delta v = f(x), \ x \in \mathbb{R}^3.
\]
This gives a way to express solutions of the Poisson equation in terms of the solution of the initial value problem for the heat equation.
(iii) Use this to obtain the expression
\[
H(x) = \frac{1}{4\pi|x|}
\]
for the fundamental solution of the Laplace equation, starting with the heat kernel in three dimensions
\[
G(t, x) = \frac{1}{(4\pi t)^{3/2}} e^{-|x-y|^2/(4t)}.
\]

Problem 2. Let the function \( g(t) \) be bounded and continuous, with \( g(0) = 0 \), and set
\[
u(t, x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-x^2/(4(t-s))} g(s)ds,
\]
defined for \( t > 0 \) and \( x > 0 \).
(i) Why does the integral above converge for all \( x > 0 \) (there is a potential problem at \( t = s \) where \( 1/(t-s)^{3/2} \) blows up)?
(ii) Verify that \( u(t, x) \) satisfies the heat equation for \( t > 0 \) and \( x > 0 \):
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.
\]
(iii) Show that \( u(t, x) \to 0 \) as \( t \to 0 \) for all \( x > 0 \).
(iv) Show that \( u(t, x) \to g(t) \) as \( x \to 0 \) for all \( t > 0 \). Hence, \( u(t, x) \) is the solution of the initial boundary value problem posed in the half-space \( x > 0 \):
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\
u(t, 0) &= g(t), \\
u(0, x) &= 0.
\end{align*}
\]
**Problem 3.** Write down an explicit formula for the solution of the initial value problem
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} + cu = 0, \quad t > 0, \quad x \in \mathbb{R} \\
u(0, x) = f(x),
\]
where \(b, c \in \mathbb{R}\) are fixed constants. Hint: use a change of variables to get rid of the term with the first derivative. Show that if \(f(x) \geq 0\) for all \(x \in \mathbb{R}^n\), then \(u(t, x) \geq 0\) for all \(t > 0\) and \(x \in \mathbb{R}^n\).

**Problem 4.** Let \(\Omega\) be a smooth bounded domain. Consider the functional
\[
I(u) = \int_{\Omega} |\nabla u(x)|^2 dx,
\]
and the admissible set
\[
\mathcal{A} = \left\{ u \in C^2(\Omega), \ u = 0 \text{ on } \partial \Omega, \ \int_{\Omega} |u(x)|^2 dx = 1 \right\}
\]
Assume that a function \(v\) is the unique minimizer of \(I(u)\) over \(\mathcal{A}\). Show that there exists a number \(\mu\) such that \(v\) satisfies
\[
-\Delta v = \mu v, \quad \text{in } \Omega, \\
v = 0, \quad \text{on } \partial \Omega.
\]
Show that \(\mu > 0\).

**Problem 5.** Let \(F(x)\) be a strictly convex function defined for \(x \in \mathbb{R}^n\) such that for any \(M > 0\) there exists \(R > 0\) so that \(F(x) > M\) if \(|x| > R\).

(i) Show that \(F\) has a unique minimum \(x_0\).

(ii) Consider an ordinary differential equation
\[
\frac{dX}{dt} = -\nabla F(X(t)), \quad X(0) = x_1,
\]
with some \(x_1 \in \mathbb{R}^n\). Show that \(G(t) = F(X(t))\) is a decreasing function of \(t\).

(iii) Show that \(X(t)\) converges to \(x_0\) as \(t \to +\infty\). This is the gradient descent method.

**Problem 6.** Let \(\Omega\) be a smooth bounded domain, and
\[
\mathcal{A} = \{ u \in C^2(\Omega) \text{such that } u = 0 \text{ on } \partial \Omega \}.
\]

(i) Fix a function \(g \in \mathcal{A}\) and define, for \(v \in \mathcal{A}\),
\[
R(v) = \int_{\Omega} gvdx.
\]
Find the function \(v\) that minimizes \(R(v)\) over
\[
\mathcal{B} = \left\{ v \in \mathcal{A} \text{ and } \int_{\Omega} |v(x)|^2 dx = 1 \right\}.
\]

**Problem 7.** Consider the functional
\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx,
\]
defined for $u \in \mathcal{A} = \{ u \in C^2(\Omega) \text{ such that } u = 0 \text{ on } \partial\Omega \}$. Fix one such $u$, and consider $G(s; u, v) = I(u + sv)$, for various $v \in \mathcal{A}$.

(i) Find a function $g$, so that for any $v \in \mathcal{A}$ we have

$$\frac{dG}{ds}(s = 0; u, v) = \int_{\Omega} g(x)v(x)dx.$$ 

(ii) Relate the heat equation

\begin{align*}
  u_t &= \Delta u, \quad \text{in } \Omega, \\
  u &= 0, \quad \text{on } \partial\Omega, \\
  u(0, x) &= f(x), \quad \text{in } \Omega
\end{align*}

to the gradient descent method for the functional $I(u)$ starting at $f$. 