Homework # 1.

1. Solve the equation
\[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \cos(x - y), \]
in two dimensions \((x, y) \in \mathbb{R}^2\), with the initial condition \(u(x, 0) = f(x)\). What would happen if we prescribe the data \(u(x, x) = f(x)\) along the line \(x = y\)?

2. Let \(u \in \mathbb{R}^n\) be a fixed vector in \(\mathbb{R}^n\), and \(c \in \mathbb{R}\) be a real number. Write down an explicit formula for a solution of the following equation
\[ \frac{\partial \phi}{\partial t} + u \cdot \nabla_x \phi + c\phi(1 - \phi) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \]
supplemented by the initial condition \(\phi(0, x) = g(x)\), where \(g(x)\) is a given smooth function. Show that the solution you found is non-negative if \(g(x) \geq 0\) for all \(x \in \mathbb{R}^n\). Is it true that \(|\phi(t, x)| \leq \sup_{x \in \mathbb{R}^n} |g(x)|\), that is, does the maximum principle hold? Next, assume that \(0 \leq g(x) \leq 1\) and show that then \(0 \leq \phi(t, x) \leq 1\).

3. Consider the PDE
\[ \frac{\partial \phi}{\partial t} + u(x) \cdot \nabla \phi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \]
with a smooth drift \(u(x)\) and the smooth initial data \(\phi(0, x) = f(x)\) that vanishes outside of a compact set.

(i) Assume that \(\text{div} u(x) = 0\) for all \(x \in \mathbb{R}^n\), and show that
\[ \int_{\mathbb{R}^n} |\phi(t, x)|^2dx = \int_{\mathbb{R}^n} f^2(x)dx. \]
Hint: multiply the PDE by \(\phi(t, x)\) and integrate by parts.

(ii) Assume again that \(\text{div} u(x) = 0\) and consider the ODE
\[ \frac{dX}{dt} = u(X), \quad X(0; x) = x. \]
Let \(S\) be a bounded open set in \(\mathbb{R}^n\) and let \(S(t) = \{X(t; x), \ x \in S\}\), that is, we solve (1) for all \(x \in S\) and consider the resulting "image" set. Show that the \(n\)-dimensional volume of \(S(t)\) is equal to that of \(S\) if \(\text{div} u(x) = 0\) for all \(x \in \mathbb{R}^n\).

(iii) Assume that \(\text{div} u \geq 1\) for all \(x \in \mathbb{R}^n\) and show that
\[ \int_{\mathbb{R}^n} |\phi(t, x)|^2dx \geq e^{t} \int_{\mathbb{R}^n} f^2(x)dx. \]

4. Let \(f(x), \ x \in \mathbb{R}\) be smooth and compactly supported function and \(k > 0\). Show that the function
\[ u(x) = \frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-y|} f(y)dy \]
solves the following ODE:
\[ -u'' + k^2u = f(x), \quad -\infty < x < +\infty. \]
This means that $\Phi(x, y) = (1/2k)e^{-k|x-y|}$ is the Green’s function for this ODE. Show that without any additional assumptions on the solutions the ODE has infinitely many solutions other than the one given above. Give a reasonable constraint on the solution that would ensure that solution is unique and prove uniqueness under that constraint.