MATH 220
solution to midterm 1

Problem 1.

(i) Define $z(s) := u(t+s,x+s)$, then

$z'(s) = 0,$
$z(-t) = u(0,x-t) = f(x-t),
\quad u(t,x) = z(0) = z(-t) = f(x-t).

(ii) First we solve the characteristic curve

\[
\begin{cases}
\frac{dX}{ds}(s; t, x) = X(s; t, x), \\
X(t; t, x) = x.
\end{cases}
\]

We have

$X(s; t, x) = xe^{s-t}.$

Next, define $z(s) := u(s, X(s; t, x))$ and we have

$z'(s) + z^2(s) = 0,$
$z(0) = u(0, xe^{-t}) = f(xe^{-t}),
\quad z(s) = \frac{f(xe^{-t})}{sf(xe^{-t}) + 1},
\quad u(t,x) = z(t) = \frac{f(xe^{-t})}{tf(xe^{-t}) + 1}.$

The solution is not defined for all $t > 0$ and for all such $f$. For example, when $f(x) \equiv -1$, we see that the solution blows up at $t = 1$ and the solution does not exist for $t \geq 1$.

(iii) First we solve the characteristic curve

\[
\begin{cases}
\frac{dX}{ds}(s; t, x) = X(s; t, x)^2, \\
X(t; t, x) = x.
\end{cases}
\]

We have

$X(s; t, x) = \frac{x}{(t-s)x + 1}.$

Note that the characteristic curve only exists for $(t,x)$ satisfying $x + \frac{1}{t} > 0$, otherwise the curve cannot cross the line $t = 0$. Thus we can only solve the equation in the domain
\{(t, x) : t > 0 \land x + \frac{1}{t} > 0\}, and only the points in this domain have characteristic curves crossing the initial value line \(t = 0\).

Define \(z(s) := u(s, X(s; t, x))\) and we have
\[
\begin{align*}
  z'(s) &= 0, \\
  z(0) &= u(0, \frac{x}{tx+1}) = f(\frac{x}{tx+1}), \\
  z(s) &= z(0), \\
  u(t, x) &= z(t) = f(\frac{x}{tx+1}).
\end{align*}
\]

As we mentioned above, for any \(t > 0\), the solution only exists for \(x > \frac{1}{t}\). Hence it is not defined everywhere.

**Problem 2.**

\[
\begin{align*}
  u(x) &= \frac{1}{2k} \left( \int_{-\infty}^{x} e^{k(y-x)} f(y) \, dy + \int_{x}^{\infty} e^{k(x-y)} f(y) \, dy \right), \\
  u'(x) &= \frac{1}{2k} \left( f(x) - k \int_{-\infty}^{x} e^{k(y-x)} f(y) \, dy - f(x) + k \int_{x}^{\infty} e^{k(x-y)} f(y) \, dy \right) \\
  &= \frac{1}{2} \left( - \int_{-\infty}^{x} e^{k(y-x)} f(y) \, dy + \int_{x}^{\infty} e^{k(x-y)} f(y) \, dy \right), \\
  u''(x) &= \frac{1}{2} \left( -f(x) + k \int_{-\infty}^{x} e^{k(y-x)} f(y) \, dy - f(x) + k \int_{x}^{\infty} e^{k(x-y)} f(y) \, dy \right) \\
  &= -f(x) + k^2 u(x).
\end{align*}
\]

Thus
\[-u''(x) + k^2 u(x) = f(x) .
\]

Note that for the corresponding homogeneous equation
\[-v''(x) + k^2 v(x) = 0,
\]
the solution has the form
\[v(x) = Ae^{kx} + Be^{-kx}.
\]

Thus the solution to the equation
\[-u''(x) + k^2 u(x) = f(x)
\]
has the form
\[u(x) = Ae^{kx} + Be^{-kx} + \frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-y|} f(y) \, dy.
\]
where $A, B$ are arbitrary numbers. So there are infinitely many solutions.

To ensure uniqueness, we need to add constraint to guarantee that $A, B$ are 0. Note that the function

$$\frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-y|} f(y) \, dy$$

is bounded because $f(x)$ is compactly supported, and $v(x) = Ae^{kx} + Be^{-kx}$ is unbounded unless both $A$ and $B$ are 0. So a reasonable constraint is, $u(x)$ is a bounded function. In other words, $u(x) = \frac{1}{2k} \int_{-\infty}^{\infty} e^{-k|x-y|} f(y) \, dy$ is the unique solution to the problem

\begin{align*}
\begin{cases}
-u''(x) + k^2 u(x) &= f(x), \\
\sup_{x \in \mathbb{R}} |u(x)| &< \infty.
\end{cases}
\end{align*}

**Problem 3.**

Define

$$\phi(r) = \frac{1}{|\partial B(x_0, r)|} \int_{\partial B(x_0, r)} u(y) \, dS(y),$$

then

$$\phi(r) = \frac{1}{|\partial B(0, 1)|} \int_{\partial B(0, 1)} u(x_0 + rz) \, dS(z),$$

$$\phi'(r) = \frac{1}{|\partial B(0, 1)|} \int_{\partial B(0, 1)} z \cdot \nabla u(x_0 + rz) \, dS(z) = \frac{1}{|\partial B(x_0, r)|} \int_{\partial B(x_0, r)} \frac{y - x_0}{r} \cdot \nabla u(y) \, dS(y)$$

$$= \frac{1}{|\partial B(x_0, r)|} \int_{\partial B(x_0, r)} \frac{\partial u}{\partial \nu}(y) \, dS(y) = \frac{1}{|\partial B(x_0, r)|} \int_{B(x_0, r)} \Delta u(y) \, dS(y) \leq 0,$$

$$\Rightarrow$$

$$u(x_0) = \phi(0) \geq \phi(r) = \phi(r) = \frac{1}{|\partial B(x_0, r)|} \int_{\partial B(x_0, r)} u(y) \, dS(y), \quad \forall 0 < r < R.$$
Since \( y \) is a minimum point we also have
\[
u(y) \leq \frac{1}{\alpha(n)r^n} \int_{B(y,r)} u(x) \, dx,
\]
which means
\[
u(y) = \frac{1}{\alpha(n)r^n} \int_{B(y,r)} u(x) \, dx,
\]
and \( u(y) \equiv u(x_0), \forall y \in B(y,r) \). Then \( y \) is an interior point of \( S \). So \( S \) is open. By the definition of \( S \) we also know that \( S \) is closed and nonempty. Then by the connectivity of \( \Omega \) we know that \( S = \Omega \), which implies \( f(x) \equiv 0, \forall x \in \Omega \). Thus \( u(x) \) is a constant function and \( f \equiv 0 \).

So if \( f \neq 0 \), then \( u(x) \) cannot attain its minimum inside \( \Omega \). This means \( u \) has to attain its minimum in \( \partial \Omega \). We also know that the boundary value of \( u \) is 0. Thus \( u(x) > 0 \) for all \( x \in \Omega \).

(ii) Prove by induction. First we have \(-\Delta u_1(x) = \lambda > 0, \forall x \in \Omega \) and \( u_1(x) = 0, \forall x \in \partial \Omega \). By (i) this means \( u_1(x) > 0, \forall x \in \Omega \), which also means \( u_1(x) > u_0(x), \forall x \in \Omega \).

For the induction step, we have
\[-\Delta(u_{n+1} - u_n) = \lambda(u_n^2 - u_{n-1}^2),\]
We see that, if \( u_n > u_{n+1} \), then the right-hand side is positive in \( \Omega \). Combining with \( u_{n+1} - u_n = 0, \forall x \in \partial \Omega \), we have that \( u_{n+1} > u_n, \forall x \in \Omega \). Thus the induction is justified. We proved that \( u_{n+1} > u_n, \forall x \in \Omega, n \in \mathbb{N} \). Notice that on the boundary \( u_n \) are always zero. So we have \( u_{n+1} \geq u_n, \forall x \in \Omega, n \in \mathbb{N} \).

Note that indeed we proved a slightly stronger statement, which is that, \{\( u_n \)\} is a strictly increasing sequence for any \( x \in \Omega \).

(iii) What we need to show is that, for sufficiently large \( \lambda \), there is no solution to the equation
\[-\Delta u = \lambda(1 + u^2)\]
with zero boundary condition. Note that if there is any solution, it has to be non-negative since the right-hand side is non-negative. We prove by contradiction.

Suppose there is some non-negative solution \( u \). Then we compare \( u \) with the function sequence \{\( u_n \)\} defined above. We have that
\[u \geq u_0,\]
\[-\Delta(u - u_{n+1}) = \lambda(u^2 - u_n^2).\]
Thus, similarly, by induction we see that \( u \geq u_n \) for any \( n \in \mathbb{N} \). If we show that the increasing sequence \{\( u_n \)\} blows up, we are done. Now let’s show this.

Consider
\[-\Delta(u_{n+1} - u_n) = \lambda(u_n^2 - u_{n-1}^2) = \lambda(u_n + u_{n-1})(u_n - u_{n-1}) \geq \lambda u_1(u_n - u_{n-1}),\]
Now for the function $u_1$, we know that it is strictly positive in $\Omega$. We choose a region $U$ that is strictly contained in $\Omega$, where “strictly contained” means that $U \subseteq \Omega$ and $\text{dist}(U, \partial \Omega) > 0$. 
(This is what we did for the Harnack’s inequality, Theorem 1.10, page 22 in lecture notes.)

On $\bar{U}$, the function $u_1$ has a positive lower bound. Let’s denote it as $C$. Then

$$u_1 \geq C > 0, \, \forall x \in U,$$

$$\begin{cases} -\Delta (u_{n+1} - u_n) \geq \lambda u_1 (u_n - u_{n-1}) \geq \lambda C (u_n - u_{n-1}), & \forall x \in U, \\ u_{n+1} - u_n \geq 0, & \forall x \in \partial U. \end{cases}$$

Now let’s consider the function sequence $v_n$ defined as

$$v_n := u_{n+1} - u_n.$$

We know that $v_n$ is a non-negative sequence. Furthermore, we will show that $v_n$ blows up to positive infinity if $\lambda$ is too big, which is sufficient to conclude that $u_n$ also blows up. To see that, we examine

$$\begin{cases} -\Delta v_n \geq \lambda Cv_{n-1}, & \forall x \in U, \\ v_n \geq 0, & \forall x \in \partial U. \end{cases}$$

Intuitively, what happens is that, when $\lambda$ is super large, the spectral radius of the operator $(-\Delta)\lambda C$ is greater than 1, which causes the iteration blowing up. To be rigorous, we use some characteristic function of $(-\Delta)$ as a lower bound for $v_0$ and we create a blow-up sequence. Now, W.L.O.G. we assume that the unit cube $D = [0, 1]^d$ is in $U$ where $d$ is the dimension (otherwise you rescale and shift the problem). The cube $D$ is nothing special here, I am just trying to give you an concrete characteristic function to convince you. We can verify that, the function

$$g(x) := \prod_{k=1}^d \sin(\pi x_k)$$

is a characteristic function of $(-\Delta)$, since

$$-\Delta g(x) = (\pi^2 d) g(x).$$

Thus, if we solve the equation

$$\begin{cases} -\Delta v(x) = g(x), & \forall x \in D, \\ v(x) = 0, & \forall x \in \partial D, \end{cases}$$

then the solution is just $v(x) = \frac{1}{\pi^2 d} g(x)$. 

Now, we consider the following function iteration on $D$

$$\begin{cases} g_0(x) = g(x), \\ -\Delta g_{n+1}(x) = (2\pi^2 d) g_n(x), & \forall x \in D, \\ g_{n+1}(x) = 0, & \forall x \in \partial D. \end{cases}$$
What would the solution sequence be? It’s not hard to see that, it is \( g_1 = 2g_0, g_2 = 4g_0, \ldots, g_n = 2^ng_0, \ldots \), which blows up when \( n \) goes to infinity. The reason is that we multiplied the eigenvalue \( \pi^2d \) by 2 as an amplifier to the iteration process.

Why I care about the function \( g \)? Now let’s go back to our iteration sequence for \( v_n \). Since \( U \) contains \( D \), so on \( D \) we have

\[
\begin{align*}
-\Delta v_n &\geq \lambda Cv_{n-1}, \quad \forall x \in D, \\
v_n &\geq 0, \quad \forall x \in \partial D.
\end{align*}
\]

Now, what happens if I choose \( \lambda \) such that \( \lambda C \geq 2\pi^2d \)? Here’s what will happen:

Since \( v_0 = u_1 - u_0 = u_1 > 0 \) in \( \bar{D} \), we know that \( v_0 \geq Eg \) for some positive constant \( E \), which means that the positive function \( v_0 \) is bounded below by the characteristic function \( g \) up to some positive constant. Now, by comparing the equation for \( v_n \) and \( g_n \), we see that \( v_n \geq Eg \), which implies that \( v_n \) is also a blow-up sequence, and so is \( u_n \). Thus proved.

(iv) We first consider the general case

\(-\Delta u = f.\)

As in the notes on page 8, let \( (X(t), Y(t)) \) be the standard random walk on \( \mathbb{Z}^2 \). Suppose we start at \( (X(0), Y(0)) = (x, y) \) in \( D \) and \( (X(N), Y(N)) = (\bar{x}, \bar{y}) \) is the first time we hit the boundary of \( D \), \( (N \) is the first step we hit the boundary, which is a random variable.) then we observe that, the function

\[ u(x, y) := \frac{1}{4} \mathbb{E}_{(x,y)} \sum_{t=0}^{N-1} f(X(t), Y(t)) \]

solves the discrete Poisson problem. (The subscript \( (x,y) \) means that the random walk starts at \( (x,y) \) and the constant \( \frac{1}{4} \) is for normalization purpose.)

An interpretation of \( u(x, y) \) is that, if we start from \( (x, y) \) and collect the value \( f/4 \) at each step along the path, then the expectation of the total value we collected before we hit the boundary is \( u(x, y) \). To see this, we use the Markov property directly. For each point \( (x, y) \) in \( D \) we have

\[ u(x, y) = \mathbb{E}_{(x,y)} \sum_{t=0}^{N-1} f(X(t), Y(t)) = \frac{1}{4} f(x, y) + \frac{1}{4} \mathbb{E}_{(x+1,y)} \sum_{t=0}^{N-1} f(X(t), Y(t)) + \mathbb{E}_{(x-1,y)} \sum_{t=0}^{N-1} f(X(t), Y(t)) + \mathbb{E}_{(x,y+1)} \sum_{t=0}^{N-1} f(X(t), Y(t)) + \mathbb{E}_{(x,y-1)} \sum_{t=0}^{N-1} f(X(t), Y(t)) \]

which means

\[ 4u(x, y) = f(x, y) + u(x + 1, y) + u(x - 1, y) + u(x, y + 1) + u(x, y - 1), \forall (x, y) \in D. \]
For $(x, y) \in \partial D$, by definition we have $u(x, y) = \frac{1}{4} \mathbb{E}_{(x, y)} \sum_{t=0}^{N-1} f(X(t), Y(t)) = 0$.

To pass to the continuous case, we consider the lattice $(h\mathbb{Z})^2$, and we define

$$u(x, y) := \frac{h^2}{4} \mathbb{E}_{(x, y)} \sum_{t=0}^{N-1} f(X(t), Y(t)),$$

then

$$\begin{cases}
    u(x, y) = \frac{1}{4} (u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h)) + \frac{h^2}{4} f(x, y), \forall (x, y) \in D, \\
    u(x, y) = 0, \forall (x, y) \in \partial D.
\end{cases}$$

Let $h \to 0$, and set $\Delta t = \frac{h^2}{4}$. The limit of the random walk is a Brownian motion. (The constant $\frac{1}{4}$ here determines how fast your Brownian motion is. It’s just some constant which has nothing to do with the big picture here.) Then we have

$$u(x, y) := \lim_{h \to 0} \frac{h^2}{4} \mathbb{E}_{(x, y)} \sum_{t=0}^{N-1} f(X(t), Y(t)) = \lim_{\Delta t \to 0} \sum_{t=0}^{N-1} f(X(t), Y(t)) \Delta t$$

$$= \mathbb{E}_{(x, y)} \int_0^T f(X(t), Y(t)) \, dt,$$

where in the last step, the symbol $(X(t), Y(t))$ is reloaded as the corresponding Brownian motion starting at $(x, y)$, and $T$ is the first time we hit the boundary.

$u(x, y)$ solves the equation

$$\begin{cases}
    -\Delta u(x, y) = f(x, y), \forall (x, y) \in D, \\
    u(x, y) = 0, \forall (x, y) \in \partial D.
\end{cases}$$

For the case where $f \equiv 1$, the solution $u$ is just the average time for such Brownian motion to hit the boundary, or in other words, the expectation of the stopping time.