Homework # 5.

1. Assume $f(z)$ is entire, $f(z+1) = f(z)$ and $|f(z)| \leq Ae^{c|z|}$ with some $c < 2\pi$. Show that $f = \text{const}$.

2. Assume that $f(z)$ is a non-constant analytic function on the open unit disk D. Suppose that for every $a \in \partial D \setminus \{1\}$ we have

$$\lim_{z \to a} |f(z)| \leq 1,$$

and for any $\delta > 0$ we have

$$\lim_{z \to 1} |1 - z|^{\delta} |f(z)| = 0.$$

Show that $|f(z)| < 1$. Construct an analytic function in D that satisfies the first condition above but not the second.

3. Assume that f is a harmonic function in the unit disk that has a smooth restriction to the boundary, and such that $f(0) = 0$ and $\nabla f(0) = 0$. Show that $f(x) = 0$ for at least four x on the boundary.

4. Let $f(z)$ and $g(z)$ be two analytic functions defined on a domain Ω with no common zeros in Ω. Assume that both $f(z)$ and $g(z)$ have only simple zeros. Prove that there are analytic functions $F(z)$ and $G(z)$ defined on Ω such that over Ω we have

$$F(z)f(z) + G(z)g(z) = 1.$$

5. Let D_0 be the punctured unit disc. We let $\mathcal{P}$ be the set of all subharmonic functions v on D_0 such that

$$\limsup_{z \to z_0} v(z) \leq 0$$

for all $z_0 \in \partial D_)$, and $limsup_{z \to 0} v(z) \leq 201314$. Prove that for any $v \in \mathcal{P}$, we have $v(z) \leq 0$ in D_0.