Homework # 4.

1. Prove that if f is holomorphic in the unit disk $\{|z| < 1\}$, and $z_1, z_2, z_n, \ldots$ are its zeros, then

$$\sum_n (1 - |z_n|) < \infty.$$

2. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha}$ is an entire function of the order $1/\alpha$.

3. Show that all conformal mappings from the upper-half plane to the unit disk are of the form

$$e^{i\theta} \frac{z - a}{z - \bar{a}},$$

with some fixed a with $\Im a > 0$.

4. Prove that $\wp''$ is a quadratic polynomial in $\wp$.

5. (i) Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(ii) Show that

$$\wp(z) = c + \pi^2 \sum_{m=-\infty}^{\infty} \frac{1}{\sin^2(\pi(z + m\tau))},$$

with some constant c.