Homework # 3.

1. Suppose f is continuous and of moderate decrease, and $\hat{f}(\xi) = 0$. Show that $f(x)$ is zero in the following way. First, fix $t \in \mathbb{R}$ and set

$$A_t(z) = \int_{-\infty}^{t} f(x)e^{-2\pi iz(x-t)}dx, \; B_t(z) = -\int_{t}^{\infty} f(x)e^{-2\pi iz(x-t)}dx.$$

Show that $A_t(x) = B_t(x)$ for all real x. Define $F(z)$ as being equal to $A_t(z)$ in the upper half-plane, and to $B_t(z)$ in the lower half-plane. Show that $F(z)$ is entire and bounded, and, finally, that $F \equiv 0$. Deduce that

$$\int_{-\infty}^{t} f(x)dx = 0,$$

for all $t \in \mathbb{R}$, and deduce that $f \equiv 0$.

2. Let $f \in L^1(\mathbb{T})$, and assume that

$$f_n = \int_0^1 f(x)e^{-2\pi inx}dx = 0,$$

for all $n \in \mathbb{Z}$. Show that $f \equiv 0$. Deduce that if α is an irrational number, and a function $f \in L^1(\mathbb{T})$ satisfies $f(x+\alpha) = f(x)$ for a.e. $x \in \mathbb{T}$, then $f \equiv$ const.

3. Let F be a holomorphic function in the right half plane that is continuous up to its boundary (the imaginary axis). Suppose that $|F(iy)| \le 1$ for all $y \in \mathbb{R}$, and, in addition, that $|F(z)| \le Ae^{|z|^\gamma}$. For which γ can you conclude that $|F(z)| \le 1$ for all z in the right half plane?

4. (a) Show the following. A function f is entire and satisfies

$$|f(z)| \le A(\varepsilon)e^{2\pi(M+\varepsilon)|z|}$$

if and only if it can be represented as

$$f(z) = \int_C e^{2\pi izw}g(w)dw,$$

where the function g is holomorphic outside the circle of radius M centered at the origin and vanishes at infinity. Here C is any circle of radius larger than M. More precisely, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$g(w) = \sum_{n=0}^{\infty} \frac{A_n}{w^{n+1}},$$

with $n!a_n = A_n(2\pi)^{n+1}$.

(b) Show that if, in addition, f and $\hat{f}$ are of moderate decrease on the real axis, then g is actually holomorphic outside the interval $[-M, M]$, and is given by

$$g(z) = \frac{1}{2\pi i} \int_{-M}^{M} \frac{\hat{f}(\xi)}{\xi - z} d\xi.$$

Show that in this case, $\hat{f}(\xi)$ is the jump of g across the real axis, that is:

$$\hat{f}(\xi) = \lim_{\varepsilon \to 0^+} (g(\xi + i\varepsilon) - g(x - i\varepsilon)).$$