Homework # 2.

1. Let $f(z)=\sum_{n=0}^\infty a_n z^n$ have radius of convergence $R=1$. Problems (a)–(c) further explore the connection between the behavior of the series and the function f at the boundary. They are completely elementary but a bit tricky.

(a) Suppose $a_n\in\mathbb{R}$ for all $n\ge 0$ and $s_n=a_0+a_1+\ldots+a_n\to\infty$ as $n\to\infty$. Prove that $f(z)$ cannot be analytically continued to any neighborhood of $z=1$.

(b) Suppose $\sum_{n=0}^\infty a_n=s$. Show that then $f(z)\to s$ as $z\to 1$ inside the region $z\in K_\alpha\cap\mathbb{D}$, $0<\alpha<\pi$ arbitrary but fixed. Here K_α is a cone with tip at $z=1$, symmetric about the x-axis, opening angle α, and with $(-\infty,1)\subset K_\alpha$ (this type of convergence $z\to 1$ is called "non-tangential convergence"). Note that $z=1$ can be replaced by any $z\in\partial\mathbb{D}$.

(c) Now assume that $na_n\to 0$ as $n\to\infty$. If $f(z)\to s$ as $z\to 1$ non-tangentially, then prove that $\sum_{n=0}^\infty a_n=s$. Note again that $z=1$ can be replaced by any $z\in\partial\mathbb{D}$.

2. (a) Let $0\le r_1<r_2\le\infty$ and suppose that u is a real-valued harmonic function on the annulus $\mathcal{A}=\{z\in\mathbb{C}\ :\ r_1<|z|<r_2\}$. Prove that there exists some unique $k\in\mathbb{R}$ and $f\in\mathcal{H}(\mathcal{A})$ such that

$$u(z)=k\log|z|+\Re f(z)\qquad\forall\, z\in\mathcal{A}$$

Next, assume that $r_1=0$. Prove that if u is bounded on $\mathcal{A}$, then $k=0$ and u extends to a harmonic function throughout $|z|<r_2$.

(b) Suppose $\Omega\subset\mathbb{C}$ is open and simply connected. Let $z_0\in\Omega$ and suppose that $u\in\Omega\setminus\{z_0\}\to\mathbb{R}$ is harmonic such that

$$u(z)-\log|z-z_0|$$

remains bounded as $z\to z_0$. Show that there exists $f\in\mathcal{H}(\Omega)$ such that $f(z_0)=0$, $u(z)=\log|f(z)|$, and f is one-to-one one some disk around z_0.

3. Suppose that u,v are harmonic in Ω so that ∇u and ∇v never vanish in Ω (we call this *non-degenerate*). If $f=u+iv$ is conformal (i.e., $f\in\mathcal{H}(\Omega)$), then we know that the level curves $u=\text{const}$ and $v=\text{const}$ in Ω are perpendicular to each other (why?). This exercise addresses the converse:

(a) Suppose v,w are harmonic and non-degenerate in Ω such that the level curves of v and w coincide in Ω. How are v and w related?

(b) Suppose u,v are harmonic and non-degenerate in Ω, and assume their level curves are perpendicular throughout Ω. Furthermore, assume that $|\nabla u(z_0)|=|\nabla v(z_0)|$ at *one point* $z_0\in\Omega$. Prove that either $u+iv$ or $u-iv$ is conformal in Ω.

4. Show that if $u_1\le u_2\le u_3\le\ldots$ are harmonic functions in Ω. Let $u=\sup_n u_n$. Then either $u\equiv\infty$ or u is harmonic in Ω.

5. Let u be sub-harmonic in the unit disk $\mathbb{D}$. Show that the following two properties are equivalent:

(i) u has a harmonic majorant on $\mathbb{D}$, i.e., there exists $h:\mathbb{D}\to\mathbb{R}$ harmonic such that $u\le h$ on $\mathbb{D}$.

$$\text{(ii)} \sup_{0<r<1} \int_0^1 u(re(\theta))\, d\theta < \infty \text{ where } e(\theta) = e^{2\pi i\theta}.$$

We say that h_0 is a *least harmonic majorant* of u iff h_0 is a harmonic majorant of u on $\mathbb{D}$ and if $h \geq h_0$ for every other harmonic majorant h of u.

Prove that if u has a harmonic majorant on $\mathbb{D}$, then it has a least harmonic majorant. Given an example of a subharmonic function u in $\mathbb{D}$ that has no harmonic majorant.

6. Let $f \in \mathcal{H}(\mathbb{D})$, $f \not\equiv 0$. Then prove that the following two properties are equivalent (here $\log^+ x = \max(0, \log x)$):

(i) $\log^+ |f|$ has a harmonic majorant in $\mathbb{D}$.

(ii) $f = \frac{g}{h}$ where $g, h \in \mathcal{H}(\mathbb{D})$ with $|g| \leq 1$, $0 < |h| \leq 1$ in $\mathbb{D}$.

7. (a) Suppose $\mathcal{Z} = \{z_n\}_{n=0}^\infty \subset \mathbb{D} \setminus \{0\}$ satisfies

$$\sum_{n=0}^\infty (1 - |z_n|) < \infty$$

Prove that

$$B(z) = \prod_{n=0}^\infty \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

converges uniformly on every $D(0, r)$ with $0 < r < 1$ to a holomorphic function $B \in \mathcal{H}(\mathbb{D})$ with $|B(z)| \leq 1$ for all $|z| < 1$. It vanishes exactly at the z_n (with the order of the zero being equal to the multiplicity of z_n in $\mathcal{Z}$).

(b) We know that $\lim_{r\to 1-} B(re^{i\theta})$ exists for almost every θ (after all, $B \in h^\infty(\mathbb{D})$). Denote these boundary values by $B(e^{i\theta})$. Prove that $|B(e^{i\theta})| = 1$ for almost every θ.