Homework # 1.

1. Assume that $f(x)$ is analytic in a domain D and that $|f(z)^2 - 1| < 1$ in D. Show that either $\mathrm{Re} f(z) > 0$ of $\mathrm{Re} f(z) < 0$ for all $z \in \Omega$.

2. If the consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|.$$

3. Find a Möbius transformation that maps the circles $\{|z| = 1\}$ and $\{|z - 1/4| = 1/4\}$ into concentric circles.

4. Prove that for a fixed w such that $|w| < 1$, the mapping

$$f(z) = \frac{w - z}{1 - \bar{w}z}$$

maps the unit disk onto itself.

5. Suppose that the function f is analytic on the unit disk $D = \{|z| < 1\}$. Show that we have

$$2|f'(0)| \leq \sup_{z,w \in D} |f(z) - f(w)|.$$

6. Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

Describe the behavior of $f(z)$ as z approaches the unit circle in the radial direction. Next, consider the function

$$g(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n},$$

with some $\alpha > 0$. Show that it can be extended continuously to the unit circle but can not be analytically continued past it. (This is much harder than the first part).