Homework # 2.

1. Let \( \mu \) be a Borel measure on \([0, 1]\) with \( \mu([0, 1]) = 1 \). Show that there exists a compact set \( K \subseteq [0, 1] \) so that \( \mu(K) = 1 \) but \( \mu(H) < 1 \) for any proper compact subset \( H \) of \( K \). \( K \) is called the support of \( \mu \). Show that every compact subset of \([0, 1]\) is the support of some Borel measure.

2. Construct a function such that each set \( \{ f(x) = \alpha \} \) is measurable for any \( \alpha \in \mathbb{R} \) but the set \( \{ f(x) > 0 \} \) is not measurable.

3. Construct a monotone function that is discontinuous on a dense set on \([0, 1]\).

4. Let \( \phi \) be a non-negative continuous function on \( \mathbb{R}^n \) such that \( \int \phi = 1 \). Given \( t > 0 \) define \( \phi_t(x) = t^{-n} \phi(x/t) \). Show that if \( g \in C^\infty(\mathbb{R}^n) \) with compact support then
   \[
   \phi_t(g) = \int_{\mathbb{R}^n} \phi_t(x)g(x)dx \to g(0).
   \]
   Because of that \( \phi_t \) is called an approximation of identity. How much can you weaken the regularity assumptions on \( \phi \) and \( g \)?

5. Let \( E_k \) be a sequence of measurable sets such that
   \[
   \sum_{k=1}^{\infty} \mu(E_k) < +\infty.
   \]
   Show that then almost all \( x \) lie in at most finitely many of the sets \( E_k \).

6. Let
   \[
   \psi(x) = \begin{cases} x, & 0 \leq x \leq 1/2, \\ 1 - x, & 1/2 \leq x \leq 1, \end{cases}
   \]
   and extend \( \psi(x) \) to a periodic function on all of \( \mathbb{R} \). Set
   \[
   f(x) = \sum_{n=1}^{\infty} \frac{1}{4^n} \psi(4^n x).
   \]
   Show that \( f(x) \) is continuous on \( \mathbb{R} \) but is nowhere differentiable.