**Problem 1:**
(i) Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$. Show that 
\[ u(x) = (f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y) \, dy \]
is bounded and continuous.
(ii) Let $E \in \mathbb{R}^n$ be a bounded measurable set. Show that 
\[ E - E = \{x - y : x, y \in E\} \]
contains an open ball centered at the origin.

(i) $|u(x)| \leq \int |f(y)| \sup |g| \, dy = ||g||_\infty \int |f(y)| \, dy = ||g||_\infty ||f||_1 < \infty$, hence $u$ is bounded.
u is continuous since for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$,
\[ |u(x + h) - u(x)| = \left| \int (f(y + h) - f(y))g(x - y) \, dy \right| \leq ||g||_\infty \int |f(y + h) - f(y)| \, dy, \]
and the integral goes to 0 as $h \to \infty$ (similar to problem 4).

(ii) We’ll prove a slightly stronger result: if $A, B$ are bounded and measurable, with $m(A \cap B) > 0$, then $A - B$ contains an open ball centered at the origin. In deed, let $f = 1_A$ and $g = 1_B$ above. Then $u(x) = m(A \cap (x + B))$, it’s continuous at $0$ and $u(0) = m(A \cap B) > 0$. This means that for $\delta > 0$ small, $x \in B_\delta(0) \implies u(x) > 0$ too. But $u(x) > 0 \implies a = x + b \in A \cap (x + B)$ for some $a \in A, b \in B$, i.e $x = a - b \in A - B$. Hence, $B_\delta(0) \subset A - B$. \(\Box\)

**Problem 2:**
Let $p \in (0, 1)$, $0 < a < p$, and suppose $A_1, \ldots, A_N$ are Lebesgue measurable subsets of $[0, 1]$ with average measure 
\[ \frac{1}{N} \sum_{i=1}^N \mu(A_i) \geq p. \]
Let 
\[ E = \{x \in [0, 1] : x \in A_i \text{ for at least } aN \text{ values of } i\}. \]
Show that $\mu(E) \geq (1 - p)/(1 - a)$.

Let $f(x) = \frac{1}{N} \sum_{i=1}^N 1_{A_i}(x)$. The number of $i$’s for which $x \in A_i$ is $Nf(x)$, therefore $E = \{x : f(x) \geq a\}$, and 
\[ \mu(E) = \mu(\{x : f(x) \geq a\}) = \mu(\{x : 1 - f(x) \leq 1 - a\}) \]
\[ \geq \frac{1}{1 - a} ||1 - f||_1 = \frac{1}{1 - a} (1 - \frac{1}{N} \sum_{i=1}^N \mu(A_i)) \leq \frac{1 - p}{1 - a}. \]
\(\Box\)

**Problem 3:**
Let $f, g \in L^1(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and assume that for all $\psi \in C^\infty(\mathbb{T})$, we have 
\[ \int_{0}^{2\pi} f(t)\psi'(t) \, dt = -\int_{0}^{2\pi} g(t)\psi(t) \, dt, \]
i.e. that $g$ is a weak derivative of $f$. Show that $f$ is absolutely continuous, and that $f' = g$ almost everywhere.
f is absolutely continuous if and only if it’s an indefinite integral with an $L^1$ integrand, and that’s what we’ll try to show, i.e. that $f(t) = \int_{0}^{t} g(s) \, ds + a$ a.e. $t$, with $a$ to be determined later. In deed, let $F(t) = f(t) - \int_{0}^{t} g(s) \, ds - a$. Then for any $C^\infty$ function $\psi$ we have
\[
\int_0^{2\pi} F(t)\psi'(t) \, dt = \int_0^{2\pi} f(t)\psi'(t) \, dt - \int_0^{2\pi} \int_0^t g(s)\psi'(t) \, ds \, dt - \int_0^{2\pi} a\psi'(t) \, dt \\
= \int_0^{2\pi} f(t)\psi'(t) \, dt - \int_0^{2\pi} g(s)(\psi(2\pi) - \psi(s)) \, ds \\
= \int_0^{2\pi} f(t)\psi'(t) \, dt + \int_0^{2\pi} g(s)\psi(s) \, ds - \psi(2\pi) \int_0^{2\pi} g(s) \, ds = 0.
\]

In the last step, \( I = 0 \) from the condition of the problem, and \( II = 0 \) by applying the same condition with \( \psi \equiv 1 \).

So \( \int_0^{2\pi} F(t)\psi'(t) \, dt = 0 \) for all \( C^\infty \) \( \psi \)'s. In general, for \( \phi \in C^\infty \) we have
\[
\phi(t) = \psi'(t) + b, \quad \text{where} \quad b = \int_0^{2\pi} \phi(s) \, ds \quad \text{and} \quad \psi(t) = \int_0^t (\phi(s) - b) \, ds \in C^\infty(\mathbb{T}),
\]
so
\[
\int_0^{2\pi} F\phi = \int_0^{2\pi} F\psi' + b \int_0^{2\pi} F = b \int_0^{2\pi} F.
\]

Now we can choose \( a \) in the definition of \( F \) so that \( \int_0^{2\pi} F = 0 \), assuring that \( \int_0^{2\pi} F\phi = 0 \) for all \( \phi \in C^\infty(\mathbb{T}) \).

The last step is to choose \( \phi \) in \( C^\infty \) with compact support in \([0, 2\pi]\) and \( \int \phi = 1 \), let \( \phi_t(x) = \frac{1}{T} \phi(\frac{x}{T}) \) on \([0, 2\pi]\), extended periodically on \( \mathbb{R} \). Then \( F * \phi_t \overset{L^1}{\to} F \), for example by homework 4 (or you could use pointwise convergence from homework 1). But \( F * \phi_t = 0 \) by what we just proved, so \( \| F \|_1 = 0 \), implying \( F = 0 \) a.e.

So \( F = 0 \) a.e., hence \( f(t) = \int_0^t g(s) \, ds + a \) a.e., and therefore it’s absolutely continuous, with \( f' = g \) a.e. \( \square \)

**Problem 4:** Let \( f \in L^1(\mathbb{R}) \). Show that
\[
\lim_{t \to 0} \int_{\mathbb{R}} |f(x+t) - f(x)| \, dx = 0.
\]

Step functions are dense in \( L^1 \), so given \( \epsilon \), pick a step function \( s \) such that \( \| f - s \|_{L^1} < \epsilon \). Because
\[
\int |f(x+t) - f(x)| \leq \int |f(x+t) - s(x+t)| + \int |s(x+t) - s(x)| + \int |f(x) - s(x)| < 2\epsilon + \int |s(x+t) - s(x)|,
\]
it’s enough to prove the statement for the step functions \( s = \sum_{i=1}^n c_i \mathbf{1}_{[a_i, b_i]} \) (\( f \in L^1 \) implies \( s \in L^1 \), so all the intervals must be bounded). \( |s(\cdot + t) - s(\cdot)| \) are dominated by \( 2 \sum_{i=1}^n |c_i| \mathbf{1}_{[a_i, b_i]} \), which is integrable, and they converge pointwise to 0 a.e., so by dominated convergence \( \int |s(x+t) - s(x)| \to 0 \) as \( t \to 0 \). \( \square \)

**Problem 5:** Suppose \( f_n(x) \geq 0 \), \( f_n \) are Lebesgue measurable on \([0, 1]\) and \( f_n(x) \to 0 \) a.e. on \([0, 1] \). Assume that
\[
\int_0^1 \phi(f_n(x)) \, dx \leq 1
\]
for some continuous function \( \phi \) such that \( \lim_{t \to \infty} \phi(t)/t = \infty \). Show that
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0.
\]
Fix $\epsilon > 0$. By Egorov’s theorem, there exists $N_\delta$ and a set $B_\delta$ with $|B_\delta| \leq \delta$ such that for all $n \geq N_\delta$, $|f_n(x)| \leq \delta$ in $[0, 1] \setminus B_\delta$, with $\delta$ to be chosen later. Then for $n \geq N_\delta$,
\[
\int_0^1 f_n \leq \delta + \int_{B_\delta} f_n = \delta + \int_{\{f_n \leq M\} \cap B_\delta} f_n + \int_{\{f_n > M\} \cap B_\delta} f_n \leq \delta + M\delta + \int_{\{f_n > M\} \cap B_\delta} f_n,
\]
for any $M$ which will be chosen independently of $\delta$ at the end. Finally for the last integral note that given $N$, there exists $M_N$ such that for all $t \geq M_N$, $\phi(t)/t \geq N$, i.e. $t \leq \phi(t)/N$. Then if $M \geq M_N$ and $f(x) \geq M \geq M_N$, $\phi(f_n(x)) \leq \phi(f_n(x))/N$, so
\[
\int_{\{f_n > M\} \cap B_\delta} f_n \leq \frac{1}{N} \int_0^1 \phi(f_n) \leq \frac{1}{N}.
\]
Hence,
\[
\int_0^1 f_n \leq \delta + M\delta + \frac{1}{N}.
\]
Now we can pick the constants $\delta = \epsilon/3$, $N$ such that $1/N \leq \epsilon/3$, and $M = \max\{1/3, M_N\}$, to ensure that for all $n \geq N_\delta$, $\int_0^1 f_n \leq \epsilon$. \hfill \Box

**Problem 6:** Let $f$ be an absolutely continuous function on $[0, 1]$. Show that if a set $A$ is measurable, then $f(A)$ is measurable. Hint: start by showing that if $E$ has Lebesgue measure 0, then $f(E)$ has Lebesgue measure 0 as well.

Let us prove first the assertion of the hint: if $m(E) = 0$, then $m(f(E)) = 0$. In deed, fixing $\epsilon$, from the absolute continuity of $f$, there exists a $\delta$ such that $\sum |y_i - x_i| \leq \delta$ implies $\sum |f(y_i) - f(x_i)| \leq \epsilon$. Now Lebesgue measure is a Radon measure, so there exists an open set $O_\delta$ with $m(O_\delta \setminus E) \leq \delta$. Write $O_\delta$ as $O_\delta = \bigcup (a_i, b_i)$ with $\sum |b_i - a_i| \leq \delta$. $f$ is continuous, so it achieves its maximum and minimum over $[a_i, b_i]$ at $c_i$ and $d_i$ respectively. This means that $f((a_i, b_i))$ is contained in the interval with endpoints $f(c_i)$ and $f(d_i)$, and $m(f((a_i, b_i))) \leq |f(c_i) - f(d_i)|$. On the other hand, $|c_i - d_i| \leq |b_i - a_i|.$

Combining the assertions above, $m(f(E)) \leq m(O_\delta) \leq \sum |f(c_i) - f(d_i)|$. But $\sum |c_i - d_i| \leq \sum |a_i - b_i|\delta$, so $\sum |f(c_i) - f(d_i)| \leq \epsilon$ from the absolute continuity of $f$. Hence, $m(f(E)) \leq \epsilon$ for all $\epsilon$, implying that $m(f(E)) = 0$.

Now $A$ is measurable, so there exist compact sets $K_i$ with $m(A \setminus K_i) \leq 1/i$. Then $A = (\bigcup_i K_i) \cup E$, where $m(E) = 0$. The image of compact sets under continuous functions is compact, so $f(K_i)$ is compact, hence measurable for all $i$'s. Also $f(E)$ is measurable since it has measure 0. Combining these we get that $f(A) = (\bigcup_i f(K_i)) \cup f(E)$ is measurable. \hfill \Box