Math 205A - Fall 2016
Homework #3
Solutions

Problem 1: (i) Show that step functions and continuous functions are dense in $L^p[0,1], 1 \leq p < \infty$. Is this true for $L^\infty$?
(ii) Show that if $f$ is integrable in $E$, then $\lim_{t \to 0} \int_E |f(x) - f(x)|dx = 0$.

(i) A step function is a function of the form $\sum_{i=1}^n c_i \chi_{A_i}$, where $A_i$’s are finitely many disjoint intervals.

Let $S$ and $C$ be the vector spaces of step functions and of continuous functions in $[0,1]$ respectively. (Note that finite linear combinations of step functions are again step functions, and the same holds true for continuous functions.) Want to show that $L^p = S$ and $L^p = C$, where closure is taken w.r.t. the $L^p$ norm.

$S = L^p$: Note that $S \subset L^p$, so $S \subset L^p$. For the reverse inclusion $L^p \subset S$, we show first that characteristic functions in $S$, then extend the result to simple functions, and finally conclude the proof using their density in $L^p$.

In deed, $\chi_A \in S$ for $A \subset [0,1]$ measurable: given $\epsilon$, $A \subset U$ for some open $U$, with $m(U-A) < \epsilon/2$. $U$ is a countable union of disjoint intervals $I_i$. Pick finitely many of those to ensure $m(U - \bigcap_1^m I_i) \leq \epsilon/2$, so $m(A - \bigcup_1^m I_i) \leq \epsilon$. Then

$$||\chi_A - \sum_1^n \chi_{I_i}||_p = m(A - \bigcup_1^m I_i)^{1/p} \leq \epsilon.$$ 
Hence, $\chi_A$ can be approximated by step functions.

The same is true for simple functions, because if $f = \sum_1^n c_i \chi_{A_i}$ for disjoint $A_i$’s and $c_i \neq 0$, we can pick finitely many of those to ensure $||f - \sum_1^n c_i \chi_{A_i}||_p < \epsilon/2$, and then approximate each $\chi_{A_i}$ by the step function $s_i$ such that $||f - s_i||_p < 2^{-(i+1)}c_i \epsilon$. $s = \sum_1^n s_i$ is again a step function, and $||f - s||_p < \epsilon$.

So simple functions are included in $S$, and since their closure in $L^p$ is $L^p$ itself, then $L^p \subset S$.

$C = L^p$: Exactly the same reasoning, up to an interpolation argument, works, because the step functions themselves can be approximated by continuous functions (in the $L^p$ norm). Indeed, given $\epsilon$ and an interval $[a, b] \subset [0, 1]$, the function $f$ that equals $0$ on $[a, b]^c$, $1$ on some $[a’, b’] \subset [a, b]$ with $m([a, b]\setminus[a’, b’]) < \epsilon$, and linear interpolation in between, satisfies $||\chi_{[a,b]} - f||_p < \epsilon$. So continuous functions are dense in the step functions, and hence, $L^p$.

Step functions are not dense in $L^\infty$: let $f = \sum_1^\infty c_i \chi_{[a_i, b_i]}$, $f \in L^\infty$, but $||f||_\infty \geq 1/2$ for any step function $s$.

Continuous functions are not dense in $L^\infty$, because $[0,1]$ is compact, and uniform ($L^\infty$) limits of uniformly continuous functions are continuous.

(ii) Note that the problem makes sense only if $f$ is defined on $E’ := E + (-\delta, \delta) : \{x + t : x \in E, t \in (-\delta, \delta)\}$ and all the $t$’s in consideration are $|t| < \delta$, so let’s extend $f$ by $0$ outside $E$. $f$ is integrable on $E’$, hence in $L^1(E’)$. Step functions are dense there, so given $\epsilon$, pick a step function on $E’$, such that $||f - s||_{L^1(E’)} < \epsilon$. Since $f = 0$ on $E \setminus E$, we can take w.l.o.g. $s = 0$ there too. Because

$$\int_E |f(x+t) - f(x)| \leq \int_E |f(x+t) - s(x+t)| + \int_E |s(x+t) - s(x)| + \int_E |f(x) - s(x)| < 2\epsilon + \int_E |s(x+t) - s(x)|,$$

it’s enough to prove the statement for the step functions $s = \sum_1^n c_i \chi_{[a_i, b_i]}$ in $E$ (if $f \in L^1$ implies $s \in L^1$, so all the intervals must be bounded). $|s(\cdot+t) - s(\cdot)|$ are dominated by $2 \sum_1^n |c_i \chi_{[a_i-\delta, b_i+\delta]}|$, which is integrable, and they converge pointwise to $0$ a.e., so by dominated convergence $\int_E |s(x+t) - s(x)| \to 0$ as $t \to 0$.

Problem 2: Show that if $\mu(E) < \infty$ and $f_n \to f$ a.e., then the followings are equivalent:
(i) $f_n$ are uniformly integrable, (ii) $\int |f_n - f| \to 0$, (iii) $\int f_n \to \int f$.

We will assume in this problem that $f$ is integrable, otherwise one can come up with counterexamples for the equivalence (for example $f_n = n\chi_{[0,1/n]} + 1/x\chi_{(1/n,1)} \to 1/x = f$ a.e., and $\int_{[0,1]} f_n \to \int_{[0,1]} f = \infty$, but $f_n$ are not uniformly integrable).

(i) $\Rightarrow$ (ii): Let $\epsilon > 0$. From uniform integrability there exists an $\alpha_\epsilon$ such that $\int_{\{|f_n| \geq \alpha_\epsilon\}} |f_n| < \epsilon/4$. Now pick $\delta < \epsilon/(4\alpha_\epsilon)$ such that $\mu(A) \leq \delta \Rightarrow \int_A |f| \leq \epsilon/4$. From Egorov’s theorem, exists a measurable set $E_\epsilon \subset E$ such that
\[ \mu(E \setminus E_r) \leq \min\{\epsilon/4, \delta\}, \text{ and } f_n \to f \text{ uniformly in } E_r. \] This implies that exists some \( N_r \) such that for all \( n > N_r, \)
\[ \int_{E_r} |f_n - f| < \epsilon/4, \text{ and therefore} \]
\[ \int_{E} |f_n - f| \leq \int_{E} |f_n - f| + \int_{E \setminus E_r} |f_n| + \int_{f_n} |f| \]
\[ \leq \epsilon/4 + \int_{\{f_n \geq \alpha_k\}} |f_n| + \int_{\{f_n < \alpha_k\} \cap (E \setminus E_r)} |f_n| + \epsilon/4 \]
\[ \leq \epsilon/2 + \alpha_k \mu(E \setminus E_r) + \epsilon/4 + \epsilon/4 \leq 3\epsilon/4 + \alpha_k \cdot \delta \leq \epsilon. \]

This proves that \( \lim f_n - f \leq \epsilon \) for all \( \epsilon \), hence (ii).

(ii) \( \implies (iii) \): \( \lim \sup |f_n| \leq \lim \sup ((|f_n - f| + |f|)) = |f| \). On the other hand, \( f \leq f \leq |f_n - f| + |f_n| \), so \( f \leq \lim \inf f_n - f + \lim \inf f_n = \lim \inf f_n \). Hence, \( \lim f_n - f \) exists and equals \( f \).

(iii) \( \implies (i) \): Suppose not. Then there exists an \( \epsilon > 0, \alpha_k \to \infty \) and a sequence of \( n_k \) such that \( \int_{|f_n| \geq \alpha_k} |f_n| \geq \epsilon \).

On the other hand, \( g_k = |f_n| \chi_{|f_n| \geq \alpha_k} \to 0 \) a.e. \((|f_n| \geq \alpha_k) \	o 0 \) because \( \int |f_n| \leq \int |f| + \delta \) after some \( N \).

\( g_k \leq f_{n_k} \) and \( f \leq f_{n_k} \to f \), so by problem 3, \( f \to 0 \), contradicting \( g_k = \int_{|f_n| \geq \alpha_k} |f_n| \geq \epsilon \).

**Problem 3:**
(i) Show that if \( |f_n| \leq g \in L^1(\Omega) \), then \( f_n \) are uniformly integrable in \( \Omega \). Does there exist a uniformly integrable family \( \{f_n\} \) with no integrable \( g \) such that \( |f_n| \leq g \)?

(ii) Let \( f_k \) and \( g_k \) be \( \mu \)-measurable, such that \( f_k \to f \mu\text{-a.e.}, \) \( g_k \to g \mu\text{-a.e.}, \) \( |f_k| \leq g_k \) and \( f_k \to f \). Show that \( \int f \to \int f \).

(i) \( \int |f_n| \geq \alpha \) \( f_n \leq \int \geq \alpha g \) since \( |f_n| \leq g \), and \( \lim_{\alpha \to \infty} \int \geq \alpha g = 0 \) since \( g \geq 0 \) is integrable.

Yes. \( f_n = \chi_{[n, n+1]} \) on \( \mathbb{R} \) are uniformly integrable (consider \( \alpha > 1 \)), and if \( f_n \leq g \), then \( g \geq 1 \) on \((0, \infty)\), which is not integrable.

(ii) Note that here we need to assume that \( g \) is integrable, otherwise the result is not true.

\( |f_n| \leq g_k \) implies \( g_k - f_k \geq 0 \) and \( g_k + f_k \geq 0 \). Apply Fatou’s lemma to both:

\[ \int (g - f) \leq \lim \inf \int (g_k - f_k) = \int g - \lim \sup \int f_k \implies \lim \sup \int f_k \leq \int f. \]

\[ \int (f + g) \leq \lim \inf \int (f_k + g_k) = \lim \inf \int f + \int g \implies \lim \inf \int f_k \geq \int f. \]

Combining these gives the result.

**Problem 4:**
(i) Show that any increasing function is a sum of an absolutely continuous and a singular function.

(ii) Does there exist a strictly increasing singular function?

(i) Let \( f \) be a monotone function. \( f' \) exists a.e., so let \( g(x) = \int_0^x f', \) and \( h = f - g \). Then \( g \) is absolutely continuous, and \( h \) is singular.

(ii) Yes. Consider the strictly increasing function \( f(x) = \sum_{q_n \in \mathbb{Q}} 2^{-n} \chi_{[q_n, \infty)} \) from HW2, and let \( h \) be its singular part. \( h \) is increasing because \( h(y) - h(x) = f(y) - f(x) - f'(x) \geq 0 \) for \( y > x \). If \( h \) wasn’t strictly increasing, then it would be constant on some interval \([x, y]\), hence continuous there, so \( f = g + h \) would also be continuous on \([x, y]\), contradicting the discontinuity of \( f \) on a dense subset of \([0, 1]\).

**Problem 5:** Construct an absolutely continuous strictly increasing function on \([0, 1]\) such that \( g' = 0 \) on a set of positive measure.

Consider the set \( E \) from HW1 with \( 0 < m(E \cap I) < |I| \) for all intervals \( I \) in \([0, 1]\), and let \( f(x) = \int_0^x \chi_E \). \( f \) is an indefinite integral, hence absolutely continuous. It is strictly increasing since for \( y > x \), \( f(y) - f(x) = m(E \cap [x, y]) > 0 \). This in turn implies that \( f' = \chi_E \), so \( f' = 0 \) on \( E^c \) with \( m(E^c) > 0 \).
**Problem 6:** Show that there exist two countable sub-collections $F_1$, $F_2$ of pairwise disjoint intervals, such that $F_1 \cup F_2$ covers $A$.

We’ll first cover $A \cap (0,1)$, then extend the argument to the whole $\mathbb{R}$, so assume for now that $A \subset (0,1)$. The strategy is to initially cover $A$ inductively by a countable collection of intervals that are not necessarily disjoint; afterwards we’ll rearrange these intervals into 2 sub-collections, each of them disjoint.

**Step 1.** Constructing a countable cover for $A$.

Let $A_1 = A$, $G_1 = \{I \in F : I \subset (0,1) \text{ and center of } I \text{ is in } A_1\}$, $\alpha_1 = \sup\{|I| : I \in G_1\} \leq 1$. If $A_1 = \emptyset$, there’s nothing to prove. Otherwise $\alpha_1 \neq 0$ because of the non-degeneracy, so choose $\bar{I}_1 \in G_1$ centered at $x_1 \in A_1$ with $|\bar{I}_1| > 3/4\alpha_1$.

Given $A_i, G_i, I_i$ for $i = 1, \ldots, n - 1$, define $A_n = A \setminus \bigcup_{i=1}^{n-1} I_i$, $G_n = \{I \in F : I \subset (0,1) \text{ and center of } I \text{ is in } A_n\}$, and $\alpha_n = \sup\{|I| : I \in G_n\}$. If $\alpha_n = 0$, then $A \subset \bigcup_{i=1}^{\infty} I_i$ (remember, the intervals are non-degenerate). Otherwise again pick $\bar{I}_n \in G_n$ centered at $x_n \in A_n$ with $|\bar{I}_n| > 3/4\alpha_n$.

First, $\alpha_n \rightarrow 0$: In deed, if $\alpha_n = 0$ for some $n$, we’re done. Otherwise $\alpha_{n+1} \leq \alpha_n$, so say $\alpha_n \downarrow \alpha \geq 0$. If $m > n$, then $x_m \notin I_n$, so $|x_m - x_n| \geq |I_n|/2 \geq 3/8\alpha_n \geq 3/8\alpha$. Therefore we have an infinite sequence $x_n$ of elements in $(0,1)$ with distance between any two $\geq 3/8\alpha$, which can only happen if $\alpha = 0$.

Now we claim that $A \subset \bigcup I_n$. If not, let $x \in A \setminus \bigcup I_n$, and $I \subset (0,1)$ any interval in $F$ centered at $x$. Since $x \in A \setminus \bigcup I_n$, then for all $n, x \in A_n$, so $I \subset G_n$, therefore $|I| \leq \alpha_n$. But $\alpha_n \rightarrow 0$, hence $|I| = 0$, contradicting the non-degeneracy assumption.

**Step 2.** Getting rid of the ‘redundant intervals’.

We’ll now get a new sub-collection $I_n'$ that has ‘less’ overlaps than the original one as follows: If $A \subset \bigcup I_n$, let $I_1' = \emptyset$, otherwise $I_1' = I_1$. In step $n$, if $A \subset (\bigcup I_1^{n-1}) \cup (\bigcup I_1^{n+1})$, let $I_n' = \emptyset$, otherwise $I_n' = I_n$. Then $A \subset \bigcup I_n'$, because by construction every point in $A$ is contained only in finitely many of the $I_n$s (if $x \in I_k$ for the first time, then $\text{dist}(x, x_k) > 0$ for $l > k$, and $|I_n| \downarrow 0$), so we could not have removed all of them.

What we achieved this way is that at most two of the non-empty $I_n'$s overlap at any point, because if $I_i, I_j, I_k$ all intersect, then one of them is included in the others, say $I_i$. But then $I_i' = \emptyset$, contradicting the non-emptiness.

**Step 3.** Obtaining $F_1$ and $F_2$.

There are many ways to do this, but one nice way is using graph theory: let each $I_n'$ be a vertex of a (possibly infinite) graph, and connect two vertexes iff the corresponding intervals overlap. By the remark above, this can have no cycles, so it’s a tree, and hence bipartite. This means that the vertices can be arranged into two sets $S_{1,2}$, each of them with no edges in between. Then put the intervals belonging to the set $S_i$ into $F_i$!

**Step 4.** Covering $A$ (not only $A \cap (0,1)$).

For each $n \in \mathbb{Z}$, pick an interval $J_n$ of radius $< 1/2$ if $n \in A$, otherwise do nothing. $\mathbb{R} \setminus \bigcup J_n$ is a disjoint union of open intervals, each $\subset (n, n+1)$ for some $n$, so pick the disjoint collections $F^n_1, F^n_2$, also disjoint from the $J_n$’s. Then $F_1 = \cup_n F^n_1 \cup \{J_n\}_n$ and $F_2 = \cup_n F^n_2$. \qed