Homework # 4.

1. Let \( f(t) \in L^p(\Omega) \), where \( \Omega \subseteq \mathbb{R}^n \). We say that
\[
\mu(t) = m\{x \in \Omega: |f(x)| > t\}
\]
is the distribution function of \(|f|\). Show that (this is a version of the Chebyshev inequality)
\[
\mu(t) \leq \frac{1}{t^p} \|f\|_{L^p(\Omega)}^p
\]
and
\[
\|f\|_{L^p(\Omega)}^p = \int_0^\infty t^{p-1} \mu(t) dt.
\]
More generally, show that we have for any differentiable increasing function \( \phi(s) \) such that \( \phi(0) = 0 \):
\[
\int_\Omega \phi(|f(x)|) dx = \int_0^\infty \phi'(\lambda) \mu(\lambda) d\lambda.
\]

2. Given two functions \( f(x) \) and \( g(x) \), \( x \in \mathbb{R}^m \), we define their convolution as
\[
f \ast g(x) = \int_{\mathbb{R}^m} f(x - y) g(y) dy.
\]
Show that if \( f \in L^p(\mathbb{R}^m) \), \( 1 \leq p < \infty \), \( \phi \geq 0 \), \( \phi \in L^1(\mathbb{R}^m) \) with \( \int \phi = 1 \) and \( \phi_t = t^{-m} \phi(x/t) \), then
\[
\lim_{t \to 0} \|\phi_t \ast f - f\|_p = 0.
\]

3. Let \( \mu \) be a positive measure on \( X \), \( \mu(X) < \infty \), \( f \in L^\infty(X; d\mu) \) and let
\[
\alpha_n = \int_X |f|^n d\mu, \quad n \in \mathbb{N}.
\]
Prove that
\[
\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.
\]

4. Let
\[
\phi_0(t) = \begin{cases} 
1, & x \in [0, 1] \\
-1, & x \in [1, 2]
\end{cases}
\]
extend it periodically to all of \( \mathbb{R} \), and define \( \phi_n(t) = \phi_0(2^n t), \ n \in \mathbb{N} \). Assume that \( \sum |c_n|^2 < \infty \) and show that the series
\[
\sum_{n=1}^\infty c_n \phi_n(t)
\]
converges for almost every \( t \).

5. Let \( B(x, \delta) \) be a ball of radius \( \delta > 0 \) centered at \( x \in \mathbb{R}^n \) and define
\[
\hat{M}f(x) = \sup_{\delta > 0} \frac{1}{\mu(B(y, \delta))} \int_{B(y, \delta)} |f(y)| dy.
\]
with the supremum taken over all balls \( B(y, \delta) \) such that \( x \in B(y, \delta) \).

(i) Use the covering lemmas to show that there exists a constant \( c > 0 \)
that depends only on dimension $n$ so that ($m$ is the $n$-dimensional Lebesgue measure)

$$m\{x : \tilde{M}f(x) > \alpha\} \leq \frac{c}{\alpha} \int_{\mathbb{R}^n} |f(y)| dy.$$ 

Now, show that if $f \in L^p(\mathbb{R}^n)$, then $\tilde{M}f \in L^p(\mathbb{R}^n)$. Hint: introduce $f_1(x) = f(x)$ if $|f(x)| > \alpha/2$ and $f_1(x) = 0$ if $|f(x)| \leq \alpha/2$ and show that $\{\tilde{M}(f) > \alpha\} \subset \{M(f_1) > \alpha/2\}$ so that

$$m(\{x : \tilde{M}f(x) > \alpha\}) \leq \frac{c}{\alpha} \int_{|f|\geq\alpha/2} |f(y)| dy.$$

Then use the relation

$$\int (\tilde{M}f)^p dy = p \int_0^{\infty} \mu(\tilde{M}f > \alpha) \alpha^{p-1} d\alpha$$

to finish the proof. This also proves that $\tilde{M}f$ is finite a.e.

(ii) Show that if $f \in L^1$ and $f \neq 0$ identically then there exist $C, R > 0$ so that $\tilde{M}f(x) \geq C|x|^{-n}$ (here $n$ is the space dimension) for all $x$ with $|x| \geq R$. Hence $m(\{x : \tilde{M}(x) > \alpha\}) \geq C'/\alpha$.

6. Set

$$Mf(x) = \sup_{\delta > 0} \frac{1}{\mu(B(x, \delta))} \int_{B(x, \delta)} |f(y)| dy.$$

with the supremum taken over all balls $B(x, \delta)$.

(i) Show that if $f \in L^1$ and $f \neq 0$ identically then there exist $C, R > 0$ so that $\tilde{M}f(x) \geq C|x|^{-n}$ (here $n$ is the space dimension) for all $x$ with $|x| \geq R$. Hence $m(\{x : M(x) > \alpha\}) \geq C'/\alpha$. Show also that $M(x) \leq \tilde{M}(x) \leq 2^n M(x)$.

(ii) Show that $|f(x)| \leq Mf(x)$ at every Lebesgue point of $f$ if $f \in L^1(\mathbb{R}^n)$. 