Math 205A - Fall 2015
Homework #3

Solutions

Problem 1: (i) Show that step functions and continuous functions are dense in $L^p[0,1], 1 \leq p < \infty$. Is this true for $L^\infty$?

(ii) Show that if $f$ is integrable in $E$, then $\lim_{t \to 0} \int_E |f(x+t) - f(x)|dx = 0$.

(i) A step function is a function of the form $\sum c_i \chi_{A_i}$, where $A_i$'s are finitely many disjoint intervals.

Let $\mathcal{S}$ and $\mathcal{C}$ be the vector spaces of step functions and of continuous functions in $[0,1]$ respectively. (Note that finite linear combinations of step functions are again step functions, and the same holds true for continuous functions.) Want to show that $L^p = \mathcal{S}$ and $L^p = \mathcal{C}$, where closure is taken w.r.t. the $L^p$ norm.

$L^p \subseteq \mathcal{S}$: First, $\chi_A \in \mathcal{S}$ for $A \subset [0,1]$ measurable: given $\epsilon$, $A \subset U$ for some open $U$, with $m(U - A) < \epsilon^p / 2$. $U$ is a countable union of disjoint intervals $I_i$. Pick finitely many of those to ensure $m(U - \bigcup_i I_i) \leq \epsilon^p / 2$, so $m(A - \bigcup_i I_i) \leq \epsilon^p$. Then $\|\chi_A - \sum_n \chi_{U_i}\|_p = m(A - \bigcup_i U_i)^{1/p} \leq \epsilon$. Hence, $\chi_A$ can be approximated by step functions.

The same is true for simple functions, because if $f = \sum c_i \chi_{A_i}$, for disjoint $A_i$'s and $c_i \neq 0$, we can pick finitely many of those to ensure $\|f - \sum c_i \chi_{A_i}\|_p < \epsilon$, and then approximate each $\chi_{A_i}$ by the step function $s_i$ such that $\|s_i - \chi_{A_i}\| < \epsilon$. $s = \sum_i s_i$ is again a step function, and $\|f - s\|_p < \epsilon$.

So simple functions are included in $\mathcal{S}$, therefore so are $L^p$ functions.

$L^p \subseteq \mathcal{C}$: Exactly the same reasoning, up to an interpolation argument, works, because the step functions themselves can be approximated by continuous functions (in the $L^p$ norm). Indeed, given $\epsilon$ and an interval $[a,b] \subset [0,1]$, the function $f$ that equals 0 on $[a,b]^c$, 1 on some $[a',b'] \subset [a,b]$ with $m([a',b'] \setminus [a,b]) < \epsilon$, and linear interpolation in between, satisfies $\|f - s\|_p < \epsilon$. So continuous functions are dense in the step functions, and hence, $L^p$.

Step functions are not dense in $L^\infty$: let $f = \sum_{n=2}^\infty \chi_{[\frac{1}{n^2}, \frac{1}{n^2}]}$. $f \in L^\infty$, but $\|f - s\|_\infty \geq 1/2$ for any step function $s$.

Continuous functions are not dense in $L^\infty$, because $[0,1]$ is compact, and uniform ($L^\infty$) limits of uniformly continuous functions are continuous.

(ii) Note that the problem makes sense only if $f$ is defined on $E' := E + (-\delta, \delta)$ and all the $t$'s in consideration are $|t| < \delta$, so let's extend $f$ by 0 outside $E$. $f$ is integrable on $E'$, hence in $L^1(E')$. Step functions are dense there, so given $\epsilon$, pick $s$ a step function on $E'$, such that $\|f - s\|_{L^1(E')} < \epsilon$. Since $f = 0$ on $E \setminus E'$, we can take w.l.o.g. $s = 0$ there too. Because

$$\int_E |f(x+t) - f(x)| \leq \int_E |f(x+t) - s(x+t)| + \int_E |s(x+t) - s(x)| + \int_E |f(x) - s(x)| < 2\epsilon + \int_E |s(x+t) - s(x)|,$$

it's enough to prove the statement for the step functions $s = \sum_i c_i \chi_{[a_i, b_i]}$ in $E$ ($f \in L^1$ implies $s \in L^1$, so all the intervals must be bounded). $\|s(\cdot + t) - s(\cdot)\|$ are dominated by $2 \sum_i |c_i|\chi_{[a_i - \delta, b_i + \delta]}$, which is integrable, and they converge pointwise to 0 a.e., so by dominated convergence $\int_E |s(x+t) - s(x)| \to 0$ as $t \to 0$. \hfill $\square$

Problem 2: Show that if $\mu(E) < \infty$ and $f_n \to f$ a.e., then the following are equivalent:

(i) $f_n$ are uniformly integrable, (ii) $\int |f_n - f| \to 0$, (iii) $\int |f_n| \to \int |f|$.

We will assume in this problem that $f$ is integrable, otherwise one can come up with counterexamples for the equivalence (for example $f_n = n\chi_{[0,1/n]} + 1/x\chi_{(1/(n+1),1]} \to 1/x = f$ a.e., and $\int_{[0,1]} |f_n| \to \int_{[0,1]} |f| = \infty$, but $f_n$ are not uniformly integrable).

(i) $\implies$ (ii): Let $\epsilon > 0$. From uniform integrability there exists an $\alpha_\epsilon$ such that $\int_{\{|f_n| \geq \alpha_\epsilon\}} |f_n| < \epsilon/4$. Now pick $\delta < \frac{\epsilon}{4\alpha_\epsilon}$ such that $\mu(A) \leq \delta \implies \int_A |f| \leq \epsilon/4$. From Egorov’s theorem, exists a measurable set $E_\epsilon \subset E$ such that
\[ \mu(E \setminus E_\epsilon) \leq \min\{\epsilon/4, \delta\} \text{, and } f_n \to f \text{ uniformly in } E_\epsilon. \] This implies that exists some \(N_\epsilon\) such that for all \(n > N_\epsilon\),
\[ \int_{E_\epsilon} |f_n - f| < \epsilon/4, \text{ and therefore} \]
\[ \int_{E_\epsilon} |f_n - f| \leq \int_{E_\epsilon} |f_n| + \int_{E_\epsilon} |f| \]
\[ \leq \frac{\epsilon}{4} + \int_{\{f_n \geq \alpha\}} f_n + \int_{\{f_n \leq \alpha\} \cap (E_\epsilon \setminus E)} |f_n| + \frac{\epsilon}{4} \leq \frac{\epsilon}{\alpha} \mu(E_\epsilon) + \frac{\epsilon}{4} + \frac{\epsilon}{\alpha} \cdot \delta \leq \epsilon. \]

This proves that \(\lim \int |f_n - f| \leq \epsilon\) for all \(\epsilon\), hence (ii).

(ii) \(\implies\) (iii): \(\limsup \int |f_n| \leq \lim \int (|f_n - f| + |f|) = \int |f|\). On the other hand, \(\int |f| \leq \int |f_n - f| + \int |f_n|\), so \(\int |f| \leq \liminf \int |f_n - f| + \liminf \int |f_n| = \liminf \int |f_n|\). Hence, \(\lim \int |f_n - f| \) exists and equals \(\int |f|\).

(iii) \(\implies\) (i): Suppose not. Then there exists an \(\epsilon > 0\), \(\alpha_k \to \infty\) and a sequence of \(n_k\)'s such that \(\int_{|f_n| \geq \alpha_k} |f_n| \geq \epsilon\).

On the other hand, \(g_k = |f_n| \chi_{|f_n| \geq \alpha_k} \to 0\) a.e. \(\chi_{|f_n| \geq \alpha_k} \to 0\) because \(\int |f_n| \leq \int |f| + \delta\) after some \(N\), \(g_k \leq |f_n|\) and \(\int |f_n| \to \int |f|\), so by problem 3, \(\int g_k \to 0\), contradicting \(\int g_k = \int_{|f_n| \geq \alpha_k} |f_n| \geq \epsilon\).

Problem 3: (i) Show that if \(|f_n| \leq g \in L^1(\Omega)\), then \(f_n\) are uniformly integrable in \(\Omega\). Does there exist a uniformly integrable family \(\{f_n\}\) with no integrable \(g\) such that \(|f_n| < g\)?
(ii) Let \(f_k\) and \(g_k\) be \(\mu\)-measurable, such that \(f_k \to f\) \(\mu\)-a.e., \(g_k \to g\) \(\mu\)-a.e., \(|f_k| \leq g_k\) and \(\int g_k \to \int g\). Show that \(\int f_k \to \int f\).

(i) \(\int_{|f_n| \geq \alpha} |f_n| \leq \int_{g \geq \alpha} g\) since \(|f_n| \leq g\), and \(\lim_{\alpha \to \infty} \int_{g \geq \alpha} g = 0\) since \(g \geq 0\) is integrable. Yes, \(f_n = \chi_{[n, n+1]}\) on \(\mathbb{R}\) are uniformly integrable (consider \(\alpha > 1\)), and if \(f_n \leq g\), then \(g \geq 1\) on \((0, \infty)\), which is not integrable.

(ii) Note that here we need to assume that \(g\) is integrable, otherwise the result is not true.
\(|f_k| \leq g_k\) implies \(g_k - f_k \geq 0\) and \(g_k + f_k \geq 0\). Apply Fatou’s lemma to both:
\(\int (g - f) \leq \liminf \int (g_k - f_k) = \int g - \limsup \int f_k \implies \limsup \int f_k \leq \int f\).
\(\int (f + g) \leq \liminf \int (f_k + g_k) = \liminf \int f + \int g \implies \liminf \int f_k \geq \int f\).

Combining these gives the result.

Problem 4: (i) Show that any increasing function is a sum of an absolutely continuous and a singular function.
(ii) Does there exist a strictly increasing singular function?

(i) Let \(f\) be a monotone function. \(f'\) exists a.e., so let \(g(x) = \int_x^f f', \) and \(h = f - g\). Then \(g\) is absolutely continuous, and \(h\) is singular.

(ii) Yes. Consider the strictly increasing function \(f(x) = \sum_{q_n \in \mathbb{Q}} 2^{-n} \chi_{(q_n, \infty)}\) from HW2, and let \(h\) be its singular part. \(h\) is increasing because \(h(y) - h(x) = f(y) - f(x) - f'_x \leq 0\) for \(y > x\). If \(h\) wasn’t strictly increasing, then it would be constant on some interval \([x, y]\), hence continuous there, so \(f = g + h\) would also be continuous on \([x, y]\), contradicting the discontinuity of \(f\) on a dense subset of \([0, 1]\).

Problem 5: Construct an absolutely continuous strictly increasing function on \([0, 1]\) such that \(g' = 0\) on a set of positive measure.

Consider the set \(E\) from HW1 with \(0 < m(E \cap I) < |I|\) for all intervals \(I\) in \([0,1]\), and let \(f(x) = \int_0^x \chi_E\). \(f\) is an indefinite integral, hence absolutely continuous. It is strictly increasing since for \(y > x\), \(f(y) - f(x) = m(E \cap [x, y]) > 0\). This in turn implies that \(f' = \chi_E\), so \(f' = 0\) on \(E^c\) with \(m(E^c) > 0\).
Problem 6: Show that there exist two countable sub-collections $F_1$, $F_2$ of pairwise disjoint intervals, such that $F_1 \cup F_2$ covers $A$.

We’ll first cover $A \cap (0,1)$, then extend the argument to the whole $\mathbb{R}$, so assume for now that $A \subset (0,1)$. The strategy is to initially cover $A$ inductively by a countable collection of intervals that are not necessarily disjoint; afterwards we’ll rearrange these intervals into 2 sub-collections, each of them disjoint.

Step 1. Constructing a countable cover for $A$.

Let $A_1 = A$, $G_1 = \{ I \in F : I \subset (0,1) \text{ and center of } I \text{ is in } A_1 \}$, $\alpha_1 = \sup\{ |I| : I \in G_1 \} \leq 1$. If $A_1 = \emptyset$, there’s nothing to prove. Otherwise $\alpha_1 \neq 0$ because of the non-degeneracy, so choose $I_1 \in G_1$ centered at $x_1 \in A_1$ with $|I_1| > 3/4\alpha_1$.

Given $A_i, G_i, I_i$ for $i = 1, \ldots, n-1$, define $A_n = A \setminus \cup_{i=1}^{n-1} I_i$, $G_n = \{ I \in F : I \subset (0,1) \text{ and center of } I \text{ is in } A_n \}$, and $\alpha_n = \sup\{ |I| : I \in G_n \}$. If $\alpha_n = 0$, then $A \subset \cup_{i=1}^{n-1} I_i$ (remember, the intervals are non-degenerate). Otherwise again pick $I_n \in G_n$ centered at $x_n \in A_n$ with $|I_n| > 3/4\alpha_n$.

First, $\alpha_n \to 0$: In deed, if $\alpha_n = 0$ for some $n$, we’re done. Otherwise $\alpha_{n+1} \leq \alpha_n$, so say $\alpha_n \downarrow \alpha \geq 0$. If $m > n$, then $x_m \notin I_n$, so $|x_m - x_n| \geq |I_n|/2 \geq 3/8\alpha_n \geq 3/8\alpha$. Therefore we have an infinite sequence $x_n$ of elements in $(0,1)$ with distance between any two $\geq 3/8\alpha$, which can only happen if $\alpha = 0$.

Now we claim that $A \subset \cup I_n$. If not, let $x \in A \setminus (\cup I_n)$, and $I \subset (0,1)$ any interval in $F$ centered at $x$. Since $x \in A \setminus (\cup I_n)$, then for all $n$, $x \in A_n$, so $I \in G_n$, therefore $|I| \leq \alpha_n$. But $\alpha_n \to 0$, hence $|I| = 0$, contradicting the non-degeneracy assumption.

Step 2. Getting rid of the ‘redundant intervals’.

We’ll now get a new sub-collection $I'_n$ that has ‘less’ overlaps than the original one as follows: If $A \subset \cup_2^\infty I_n$, let $I'_1 = \emptyset$, otherwise $I'_1 = I_1$. In step $n$, if $A \subset (\cup_1^{n-1} I'_l) \cup ((\cup_1^{n+1} I_l))$, let $I'_n = \emptyset$, otherwise $I'_n = I_n$. Then $A \subset \cup I'_n$, because by construction every point in $A$ is contained only in finitely many of the $I_n$’s (if $x \in I_k$ for the first time, then $dist(x, x_l) > 0$ for $l > k$, and $|I_n| \downarrow 0$), so we could not have removed all of them.

What we achieved this way is that at most two of the non-empty $I'_n$’s overlap at any point, because if $I_i, I_j, I_k$ all intersect, then one of them is included in the others, say $I_i$. But then $I'_i = \emptyset$, contradicting the non-emptiness.

Step 3. Obtaining $F_1$ and $F_2$.

There are many ways to do this, but one nice way is using graph theory: let each $I'_n$ be a vertex of a (possibly infinite) graph, and connect two vertices iff the corresponding intervals overlap. By the remark above, this can have no cycles, so it’s a tree, and hence bipartite. This means that the vertices can be arranged into two sets $S_{1,2}$, each of them with no edges in between. Then put the intervals belonging to the set $S_i$ into $F_i$!

Step 4. Covering $A$ (not only $A \cap (0,1)$).

For each $n \in \mathbb{Z}$, pick an interval $J_n$ of radius $1/2$ if $n \in A$, otherwise do nothing. $\mathbb{R} \setminus (\cup J_n)$ is a disjoint union of open intervals, each \subset (n, n+1) for some $n$, so pick the disjoint collections $F^n_1, F^n_2$, also disjoint from the $J_n$’s. Then $F_1 = \cup_n F^n_1 \cup \{ J_n \}$ and $F_2 = \cup_n F^n_2$. □