Solution Set
Math 205a - Fall 2011
MIDTERM

Problem 1 Let f_n be a sequence of functions in $L^2([0,1])$ such that $\|f_n\|_{L^2} \leq 1$ for all n. Show that for any $\epsilon > 0$ there exists N so that for all $n \geq N$ we have

$$m\{x \in [0,1] : |f_n(x)| \leq n^{2/3} \text{ for all } n \geq N\} \geq 1 - \epsilon$$

Here m is the Lebesgue measure and the L^2-space is with respect to the Lebesgue measure as well. Hint: estimate the measure of the set $\{x \in [0,1] : |f_n(x)| \geq n^{2/3}\}$ for one n.

Let $B_n = \{x : |f_n(x)| \geq n^{2/3}\}$. Then we have that

$$1 \geq \int_{B_n} |f_n|^2 \geq m(B_n) n^{4/3}$$

Hence we have that $m(B_n) \leq n^{-4/3}$. Notice that $\{x : |f_n(x)| \leq n^{2/3}, \text{ for all } n \geq N\} = \cap_{n=N}^{\infty} B_n^c$. Hence we have that

$$m(\{x : |f_n(x)| \leq n^{2/3} \text{ for all } n \geq N\}) = 1 - m(\bigcup_{n=N}^{\infty} B_n) \leq 1 - \sum_{n=N}^{\infty} n^{-4/3}$$

Choosing N large enough finishes the proof. ∎

Problem 2 Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{t \to 0} \int_{\mathbb{R}} |f(x+t) - f(x)| dx = 0$$

Let $(S_t f)(x) = f(x+t)$ be the shift operator. Assume that $f \in C_c(\mathbb{R})$. Let $K = [-N, N]$ contain its support, let $K_1 = [-N-1, N+1]$, and fix $\epsilon > 0$. Choose $\delta > 0$ such that $|S_t f - f| \leq \frac{\epsilon}{2N+2}$. Then notice that if $t < \min\{\delta, 1\}$ we have that

$$\begin{aligned} \|S_t f - f\| &= \int_{K_1} |S_t f - f| dx \\ &\leq \int_{K_1} \frac{\epsilon}{\mu(K_1)} \\ &= \epsilon \end{aligned}$$

Hence $\lim_t \|S_t f - f\| = 0$.

Fix $\epsilon > 0$. Now for general $f \in L^1$, find $g \in C_c(\mathbb{R})$ such that $\|g - f\| \leq \epsilon/3$. Then we have that

$$\limsup_{t \to 0} \|S_t f - f\| \leq \limsup_{t \to 0} \|S_t g - S_t f\| + \limsup_{t \to 0} \|S_t g - g\| + \limsup_{t \to 0} \|g - f\| \leq \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} < \epsilon$$

Since this is true for all $\epsilon > 0$ then we must have that $\lim \|S_t f - f\| = 0$. ∎

Problem 3 Suppose that $f_n(x) \geq 0$, f_n are Lebesgue measurable on $[0,1]$ and $f_n(x) \to 0$ a.e on $[0,1]$. Assume that

$$\int_0^1 \phi(f_n(x))dx \leq 1$$

for some continuous function $\phi(t)$ such that $\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty$. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

Fix $\epsilon > 0$. Find C such that for all $t \geq C$ we have that $\frac{\phi(t)}{t} \geq 3/\epsilon$. Use Egorov's theorem to find G where $f_n \to 0$ uniformly on G and where $m(G) \geq 1-\delta$. Choose $\delta \leq \epsilon/3$. Now choose N such that $n \geq N$ implies that $f_n \leq \epsilon/(3C)$ on G. Let $M_n = \{x : f_n(x) \geq C\}$.

$$\begin{aligned}\int_0^1 f_n(x)dx &\leq \int_G \epsilon/3 + \int_{G^c \cap M_n} f_n + \int_{G^c \setminus M_n} f_n \\ &\leq \frac{\epsilon}{3} + \int \frac{\epsilon}{3}\phi(f_n(x)) + C\mu(G^c) \\ &\leq \epsilon\end{aligned}$$

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Problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be nowhere continuous. Prove that there exists $\epsilon > 0$ and an interval (a,b) so that

$$\limsup_{y\to x} f(y) \liminf_{y\to x} f(y)$$

for all $x \in (a,b)$.

Let $M_n = \{x : \limsup f(y) - \liminf f(y) \geq 1/n\}$. Since f is nowhere continuous then we have that $\mathbb{R} = \cup M_n$. Hence it suffices to show that M_n are closed. If we do this then the Baire category theorem implies that there exists n such that M_n contains an interval. This is immediate once we show that $g(x) = \limsup_{y\to x} f(y)$ and $h(x) = \limsup_{y\to x} -f(y)$ are upper semicontinuous. We'll show that g is upper semicontinuous but similarly for h. Fix $\alpha > 0$ and find x such that $g(x) < \alpha$. Choose $\epsilon > 0$ such that $g(x) + \epsilon < \alpha$. Then, by definition of g, there exists $r > 0$ such that for all $y \in B_r(x)$ we have that $f(y) < g(x) + \epsilon$. Then if $y \in B_r(x)$, it follows that $g(y) \leq g(x) + \epsilon < \alpha$. Hence $\{x : g(x) < \alpha\}$ is an open set and so g is upper semicontinuous. ■

Problem 5 Suppose f_n is a sequence of absolutely continuous functions on $[0,1]$ such that

$$\int_0^1 f_n dx = 0$$

and

$$\int_0^1 |f_n| p dx \leq 1$$

for some $p > 1$. Show that f_n has a uniformly converging subsequence.

First notice that for every n, we have that, for $y < x$,

$$\begin{aligned}|f_n(x) - f_n(y)| &= |\int_y^x f_n'(s)dx| \\ &\leq \|1\|_{L^q(y,x)} \|f_n'\|_{L^p(y,x)} \\ &\leq |x-y|^{1/q}\end{aligned}$$

Also notice that since $\int_0^1 f_n(x)dx = 0$ for every n then there exists x_n such that $f_n(x_n) = 0$. Hence we have that $|f_n(x)| \leq 1$ for every n and every x. At this point, if we knew it, we could apply the Arzela-Ascoli theorem. Since we haven't covered it, let's do this by hand!

Let $D_n = \{\frac{k}{2^n} : k \in \{0, 1, \ldots, 2^n\}\}$. Let $n_1(k)$ be a subsequence such that $f_{n_1(k)}(x)$ converges on D_1, which is possible because D_n is finite. Given a subsequence $n_m(k)$ of $n_{m-1}(k)$ such that $f_{n_m(k)}$ converges on D_m, we can also get a subsequence $n_{m+1}(k)$ of $n_m(k)$ such that $f_{n_{m+1}(k)}$ converges on D_{m+1}. Define $n(k) = n_k(k)$. Notice that this subsequence converges uniformly on D_m for every m.

We'll show that $f_{n(k)}$ is uniformly Cauchy, finishing the problem. Fix $\epsilon > 0$. Find m large enough that $x \in [0, 1]$ implies that $d(x, D_m) \leq (\epsilon/5)^q$. Find N large enough that $k, l \geq N$ implies that $|f_{n(k)}(x) - f_{n(l)}(x)| \leq \epsilon/5$ for all $x \in D_m$. Now fix $x \in [0, 1]$ and let $y \in D_m$ be the closest element to x. Then we have that, for $l, k \geq N$

$$\begin{aligned} |f_{n(k)}(x) - f_{n(l)}(x)| &\leq |f_{n(k)}(x) - f_{n(k)}(y)| + |f_{n(k)}(y) - f_{n(l)}(y)| + |f_{n(l)}(y) - f_{n(l)}(x)| \\ &\leq \epsilon/5 + \epsilon/5 + \epsilon/5 \\ &< \epsilon \end{aligned}$$

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