Math 205A Problem Set 5

Solutions

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1 Problem 1

(i) For any closed ball $B = B(y, \delta)$ that contains x the ball $B(x, 2\delta)$ contains B and has volume $2^n m(B)$. Thus, for any x in the set $\{x : \tilde{M}f(x) > \alpha\}$ we can find a closed ball B around it such that $\frac{1}{m(B)} \int_B |f| \geq \frac{\alpha}{2^n}$. By Besikovich's theorem there are $N(n)$ disjoint collections of such balls that cover $\{x : \tilde{M}f(x) > \alpha\}$ so that at least one of them covers a subset of measure at least $\frac{1}{N(n)} m(\{x : \tilde{M}f(x) > \alpha\})$. Thus, $m(\{x : \tilde{M}f(x) < \alpha\}) \leq \frac{2^n N(n)}{\alpha} \displaystyle\int |f|$ as claimed.

Now let f_1 be as in the hint. Then, $f_1(x) \geq |f(x)| - \alpha/2$ and therefore $\tilde{M}f_1 \geq \tilde{M}f - \alpha/2$. Then $m(\{x : \tilde{M}f(x) > \alpha\}) \leq m(\{x : \tilde{M}f_1(x) > \alpha/2\}) \leq \frac{c}{\alpha} \int |f_1| = \frac{2c}{\alpha} \int_{|f| > \alpha/2} |f|$. Now just apply problem 3 of HW4 to get $\displaystyle\int (\tilde{M}f)^p = p \int_0^\infty m(x : \tilde{M}(f(x) > \alpha)\alpha^{p-1} d\alpha \leq$ $\displaystyle 2cp \int_0^\infty \left(\int_{|f| > \alpha/2} |f| \right) \alpha^{p-2} d\alpha \leq 2cp ||f||_{L^p} \int_0^\infty \alpha^{-2} d\alpha < \infty$

(ii) Chose a ball $B = B(0, r)$ such that $\int_B |f| = C > 0$ (it exists if f is not the zero function). For any x with $|x| > r$ we have $B \subset B(x, 2|x|)$, thus $\tilde{M}f(x) \geq \frac{\displaystyle\int_B |f|}{m(B(x,2|x|)} = \frac{C}{c_1 2^n |x|^n}$ where c_1 is the volume of the unit ball.

2 Problem 2

(i) Both M and $\tilde{M}$ maximise the same quantity, $\tilde{M}$ is defined over a bigger set, i.e. $\tilde{M} \geq M$ is immediate. If $B = B(y, \delta)$ contains x then $B \subset B(x, 2\delta)$. Taking supremums and using that $\mu(B(x, 2\delta)) = 2^n \mu(B(x, \delta))$ we get the second inequality.

(ii) For all $\epsilon > 0$ we can find $\delta > 0$ such that $\frac{1}{\mu(B(x,\delta))} \displaystyle\int_{B(x,\delta)} |f(x) - f(y)| dy < \epsilon$. Now, by the triangle inequality we have $|f(x)| \leq |f(x) - f(y)| + |f(y)|$. Integrating this inequality over $B(x, \delta)$ and using the above inequality we get $\frac{1}{\mu(B(x,\delta))} \displaystyle\int_{B(x,\delta)} |f(x)| dy = |f(x)| <$ $\displaystyle \epsilon + \frac{1}{\mu(B(x, \delta))} \int_{B(x,\delta)} |f(y)| dy \leq \epsilon + Mf(x)$. Since $\epsilon > 0$ was arbitrary the result follows.

3 Problem 3

Suppose $f \in L^1$, i.e. $\int |f| = M < \infty$. We have $m(\lambda) = \lambda.\mu(x : |f(x)| > \lambda) \leq \int_{|f|>\lambda} |f| \leq \int |f| = M$, i.e. $m(\lambda) \leq M$ for all λ.
Consider the function $f(x) = \frac{1}{x}$ defined over $(0,1)$. Then clearly $m(\lambda) = 1$ for all λ but $\int_0^1 f(x)dx = \infty$.

4 Problem 4

(i) Since f is continuous and compactly supported it is bounded. Suppose $|f| < M$. If $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ then $f(mx) = 0$ for $m > n$, i.e. for all such x we have $h_c(x) \leq n^c M$. Summing up over all the intervals we get $\int_R |h_c(x)| \leq \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) n^c M < M \sum_{n=1}^{\infty} n^{c-2} < \infty$.
(ii) As in part (i) we have $h_1(x) \leq nM$ when $x > \frac{1}{n+1}$. If $\lambda > M$ then $nM \leq \lambda < (n+1)M$ for some $n > 0$. We have $m(\lambda) = \lambda.\mu(|h_1| > \lambda) \leq \lambda(|h_1| > nM) \leq \lambda.\frac{1}{n+1} < \frac{(n+1)M}{n+1} = M$
If $\lambda < M$ then we use the fact that h_1 is supported on $[0, 1]$ to get $m(\lambda) \leq M.1. = M$.
To show that $h_1 \notin L^1$ we use the fact that the strictly positive function f attains its minimum $P > 0$ on the interval $[1/4, 3/4]$. Then for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ we know that $f(mx) \geq P$ for $m = [(n + 1)/2]$, thus $h_1(x) \geq mP$. This gives $\int_0^{1/2} |h_1| \geq \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \frac{n+1}{2} P > P \sum_{n=2}^{\infty} \frac{1}{2n} = \infty$.
(iii) We have $h_c(x) \geq m^c P$ when $x < 1/n$ where $m = [(n + 1)/2]$. Therefore $\mu(x : |h_c| > \frac{m^c P}{2}) \geq \frac{1}{n}$, so for $\lambda = m^c P/2$ we have $m(\lambda) \geq \frac{P}{2}[\frac{n+1}{2}]^c \frac{1}{n} \geq \frac{P}{2^{c+1}} n^{c-1}$ which is unbounded as $n \to \infty$.

5 Problem 5

Any function on X is bounded as X is finite. Thus $L^\infty(X)$ is the set of all functions which is two-dimensional space spanned by $f : X \to R$, $f(a) = 0, f(b) = 1$ and $g : X \to R$, $g(a) = 1, g(b) = 0$. Since $\mu(b) = \infty$ any function h with $h(b) \neq 0$ is not integrable. Thus L^1 is one-dimensional space spanned by g. Taking duals preserves the dimension of a finite dimensional space, therefore $L^\infty \neq (L^1)^*$

6 Problem 6

We have $\int_0^1 f(x)a(nx)dx = \int_0^n f\left(\frac{x}{n}\right)a(x)\frac{dx}{n} = \frac{1}{n}\sum_{j=0}^{n-1}\int_0^1 f\left(\frac{x+j}{n}\right)a(x)dx = \int_0^1 \frac{\sum_{j=0}^{n-1} f\left(\frac{x+j}{n}\right)}{n}a(x)dx.$

Since f is Riemann integrable for all $\epsilon > 0$ we can chose N such that for $n > N$ we have $\left|\frac{\sum\limits_{j=0}^{n-1} f(\frac{x+j}{n})}{n} - \int_0^1 f(x)dx\right| < \epsilon$. Then $\left|\int_0^1 f(x)a(nx)dx - \int_0^1 f(y)dy \int_0^1 a(x)dx\right| \leq \epsilon ||a||_{L^1}$. Thus, $\int_0^1 f(x)a(nx)dx$ converges to $\int_0^1 f(y)dy \int_0^1 a(x)dx$.