Solution Set
Math 205a - Fall 2011
Problem Set 4

Problem 1 Let $g \in L^q[0,1]$ and define a mapping $G : L^p \to \mathbb{R}$ by $G(f) = \int_0^1 fg dx$. Show that G is a bounded linear functional on L^p.

Linearity follows from the linearity of integrals. Boundedness follows from Hölder's inequality (let $C = \|g\|_q$). ∎

Problem 2 Let f_n be a sequence of functions in L^p, $1 < p < \infty$, which converge almost everywhere to $f \in L^p$. Suppose that $\|f_n\|_p \leq M$ and show that for all $g \in L^q$ we have that $\int f_n g \to \int fg$. Is this true if $p = 1$ or $p = \infty$?

Without loss of generality assume that $M = 1$ and notice that $\|f\|_p \leq \liminf \|f_n\|_p \leq 1$. Fix $\epsilon > 0$. Let h be a function of compact support, K, such that $\|g - h\|_q \leq \epsilon$. Find $\delta > 0$ such that $\int_A |h|^q \leq \epsilon^q$ for all sets $\mu(A) < \delta$. Let $G \subset K$ be a set such that $f_n \to f$ uniformly on G and such that $\mu(B) < \delta$, where $B = K \setminus G$. Then we have that, for n large enough that $|f_n - f| < \epsilon$ on G,

$$\begin{aligned}
\|g(f - f_n)\|_1 &\leq \|h(f - f_n)\|_1 + \|(g - h)(f - f_n)\|_1 \\
&\leq \int_G |h(f - f_n)| + \int_B |h(f - f_n)| + \|g - h\|_p \|f - f_n\|_q \\
&\leq \int_G |h|\epsilon + \left(\int_B |h|^q\right)^{1/q} \left(\int_B |f - f_n|^p\right)^{1/p} + 2\epsilon \\
&\leq \|h\|_p \epsilon + 2\epsilon + 2\epsilon \\
&\leq (\|g\|_p + 5 + \epsilon)\epsilon
\end{aligned}$$

Here the last step is a result of $\|h\|_p \leq \|g - h\|_p + \|g\|_p$.

For $p = 1$, this is not true. Take for example $f_n = 1_{[n,n+1]}$, $f = 0$, and $g = 1$. For $p = \infty$, this is true. Notice that $|g(f - f_n)| \leq 2M|g|$. Hence the dominated convergence theorem gives us the result. ∎

Problem 3 Let $f(t) \in L^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$. Show that $\mu(t) \leq t^{-p}\|f\|_p^p$ and $\|f\|_p^p = p \int_0^\infty t^{p-1}\mu(t)dt$. More generally, if ϕ is an absolutely continuous increasing function such that $\phi(0) = 0$ then we have that

$$\int_\Omega \phi(|f(x)|)dx = \int_0^\infty \phi'(\lambda)\mu(\lambda)d\lambda$$

To see the inequality:

$$\begin{aligned}
\mu(t) &= \int_\Omega 1_{\{|f| \geq t\}} \\
&\leq \int_{\{|f| \geq t\}} \frac{|f|^p}{t^p} \\
&\leq \int_\Omega \frac{|f|^p}{t^p}
\end{aligned}$$

Now, notice that the first fact about μ follows from the more general one. Hence let ϕ be an abitrary absolutely continuous increasing function with $\phi(0) = 0$. Then we have, using problem 7,

$$\begin{aligned}\int_\Omega \phi(|f(x)|)dx &= \int_\Omega \int_0^{|f(x)|} \phi'(y)dydx \\ &= \int_\Omega \int_0^\infty \phi'(y)1_{[0,|f(x)|]}(y)dydx \\ &= \int_0^\infty \int_\Omega \phi'(y)1_{[y,\infty]}(|f(x)|)dydx \\ &= \int_0^\infty \int_\Omega \phi'(y)1_{[y,\infty]}(|f(x)|)dxdy \\ &= \int_0^\infty \phi'(y)\mu(y)dy\end{aligned}$$

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Problem 4 Show that if $f \in L^p$, $1 \le p < \infty$, $\phi \ge 0$, $\phi \in L^1$ with $\int \phi = 1$ then

$$\lim_{t\to 0} \|\phi_t * f - f\|_p = 0$$

where $\phi_t = t^{-1}\phi(x/t)$.

We will use the following inequalities. First, if $f \in L^p$ and our measure space has measure 1 then we have that

$$\|f\|_1 \le \|f\|_p \cdot \|1\|_q = \left(\int |f|^p\right)^{1/p}$$

Hence we have that $\left(\int |f|\right)^p \le \int |f|^p$. Next notice that if $f \in L^p$ and $g \in L^1$ then we have that

$$\begin{aligned}\|g * f\|_p &= \int \left|\int g(x)f(y-x)dx\right|^p dy \\ &\le \int\int \frac{|g(x)|}{\|g\|_1}\|g\|_1^p |f(y-x)|^p dxdy \\ &= \|g\|_1^p \int \frac{|g(x)|}{\|g\|_1} \int |f(y-x)|^p dydx \\ &= \|g\|_1^p \|f\|_p^p\end{aligned}$$

Here the first inequality follows from above, using integration against $\frac{|g(x)|}{\|g\|_1}$ as the measure.

Suppose that $g \in C_c(\mathbb{R})$. Choose N large enough that $g = 0$ outside of $[-N, N]$. Let $M = \max|g(x)|$. Fix $\epsilon > 0$. Choose K large enough that $\int_{[-K,K]} \phi \ge 1 - \epsilon$. Find $\delta > 0$ such that if $t \le K\delta \le 1$ then we have that $|g(x - ty) - g(x)|^p \le \epsilon$. Then we have that if $t \le K\delta$ we have that

$$\begin{aligned}\|\phi_t * g - g\|_p^p &= \int \left|\int \phi_t(y)(g(x-y) - g(x))dy\right|^p dx \\ &\le \int\int |\phi_t(y)||g(x-y) - g(x)|^p dydx \\ &= \int\int |\phi(y)||g(x-ty) - g(x)|^p dydx \\ &= \int_{[-K,K]} \phi(y) \int_{[-N-1,N+1]} |g(x-ty) - g(x)|^p dxdy + \int_{[-K,K]^c} \phi(y) \int |g(x-ty) - g(x)|^p dxdy \\ &= \int_{[-K,K]} \phi(y) 2(N+1)\epsilon dy + \int_{[-K,K]^c} \phi(y) 2^p \|g\|_p^p dy \\ &= \epsilon(2(N+1) + 2^p\|g\|_p^p)\end{aligned}$$

Here the first inequality follows from above, taking our measure to be integrating against ϕ. Hence this holds for all functions $g \in C_c(\mathbb{R})$.

Now, fix $\epsilon > 0$ and find $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$. Then we have that

$$\|\phi_t * f - f\|_p \leq \|\phi_t * (f - g)\|_p + \|\phi_t * g - g\|_p + \|f - g\|_p$$

From above, two of these are less than ϵ and the third goes to 0 as $t \to 0$. This finishes the problem. ∎

Problem 5 Let μ be a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(X; d\mu)$ and let $\alpha_n = \int_X |f|^n d\mu$. Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$$

Assume, for ease of computation, that $\mu(X) = 1$.

First notice that if we replace f by $g = f/\|f\|_\infty$ and we prove that $\lim \frac{\int |g|^{n+1}}{\int |g|^n} = 1$, then the problem follows. Hence, assume WLOG that $\|f\|_\infty = 1$. We claim that this sequence is increasing. This is the same as showing that $\alpha_{n+1}^2 \leq \alpha_n \alpha_{n+2}$. This follows direction from Hölder's inequality.

$$\alpha_{n+1}^2 \leq \left[\alpha_n^{1/2} \alpha_{n+2}^{1/2}\right]^2$$

where we use $p = 1/2$ and $q = 1/2$ for the functions $|f|^{n/2}$ and $|f|^{(n+2)/2}$. Since this sequence is increasing and since $\alpha_{n+1}/\alpha_n \leq 1$, then it follows that it is convergent. For sequences where this limit exists, we know that $\lim \alpha_{n+1}/\alpha_n = \lim \alpha_n^{1/n}$. Hence we will show that the latter value converges to 1.

Fix $\epsilon > 0$ and let $A = \{x : |f(x)| > 1 - \epsilon\}$. Then we have that

$$(\alpha_n)^{1/n} \geq \left(\int_A (1-\epsilon)^n\right)^{1/n} = (1-\epsilon)\mu(A)^{1/n} \to 1 - \epsilon$$

Since clearly $\alpha_n^{1/n} \leq 1$ for all n, then we have that $1 - \epsilon \leq \lim \alpha_n^{1/n} \leq 1$. Letting ϵ tend to zero finishes the proof. ∎

Problem 6 Let $f \geq 0$ be a measurable function on $X \times Y$. Show that then (i) for almost all x the function $f(x, y)$ is ν-measurable (as a function of y with x being a parameter) and for almost all y the function $f(x, y)$ is μ-measurable (as a function of x with y being a parameter); (ii) $\int f(x, y) d\mu(x)$ is measurable and $\int f(x, y) d\nu(y)$ is measurable; (iii) $\int_{X\times Y} f d(\mu \times \nu) = \int_X \left[\int_Y f d\nu\right] d\mu = \int_Y \left[\int_X f d\mu\right] d\nu$; (iv) Show that the assumption that f is non-negative may not be removed: take $X = Y = \mathbb{N}$, let $\mu = \nu$ be the counting measure and let

$$f(n, m) = \begin{cases} 1, & n = m \\ -1, & n = m + 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that $\int |f| d(\mu \times \nu) = \infty$ while $\int(\int f d\mu) d\nu$ and $\int(\int f d\nu) d\mu$ both exist and are not equal.

For (i), (ii), (iii), notice that this is essentially the method of proof that the notes use. Hence refer to the notes for those parts of the problem.

For (iv): Notice that $|f(n, m)| = 1$ whenever $n = m$ or $n = m + 1$. This happens on a set of measure ∞ so $\int |f| d(\mu \times \nu) = \infty$. On the other hand, $\int f(n, m) d\mu(n) = 0$ and $\int f(n, m) d\nu(m) = 0$ for all $n \geq 2$. $\int f(1, m) d\nu(m) = 1$. Hence we have that $\int(\int f d\mu) d\nu = 0$ but $\int(\int f d\nu) d\mu = 1$. ∎

Problem 7 Prove the Besikovitch theorem in the one-dimensional case by hand. What is the smallest number of disjoint collections that you will need?

The smallest number of disjoint collections is 2. To see that one is insufficient notice that we cannot cover the set of centers of $\{[0,1],[1,2]\}$ with one disjoint collection from this set.

Now, assume WLOG that $D < 1$. We'll do this this for $A_n = [n, n+1] \cap A$, then we'll notice that it can extend. Find $B \in \mathcal{F}_{-1}$ such that $|B| \geq (3/4)\sup\{|B'| : B' \in \mathcal{F}_{-1}\}$ where $\mathcal{F}_{-1} = \{B' \in \mathcal{F} :$ the midpoint of B' is contained in $A_n\}$. Call B B_0. Given B_n choose B_{n+1} as follows. Let $d_n = \{|B| : B \in \mathcal{F}_n\}$ where $\mathcal{F}_n = \{B' \in \mathcal{F} :$ the midpoint of B' is in $A_n \setminus (\cup_{i=1}^{n-1} B_i)\}$. Choose B_{n+1} such that $|B_{n+1}| \geq (3/4)d_n$ and such that $B_{n+1} \in \mathcal{F}_n$.

This collection is either finite or infinite. The first case is trivial to deal with so assume from now on that it is infinite. Now if, for all n, $|B_n| \geq \epsilon'$ for some $\epsilon' > 0$ then we would have that there exists $\epsilon > 0$ such that for all n, $d_n \geq \epsilon$. However, as a result of our midpoint condition we can see that the union of any balls must have measure greater than or equal to $k\frac{\epsilon}{2}$. Choosing k large enough shows this is absurd. Hence we have that $|B_n| \to 0$.

Now we claim that B_n can be separated into two disjoint collections. To see this suppose there are $k < m < n$ such that $B_k \cap B_m \cap B_n \neq \emptyset$. Then this intersection must contain the midpoint of one of these balls. By construction it must be the midpoint of B_k. Now there are two situations, either the midpoints of both B_n and B_m lie, WLOG, to the right of B_k or they lie on opposite sides of B_k. The latter case is not a problem since then we may remove B_k. In the former case we notice that then, since the midpoint of B_n is not an element of B_m, that the rightmost half of B_m and the rightmost half of B_k are disjoint and contained in the leftmost half of B_n. Hence we have that $|B_n| \geq |B_m| + |B_k| \geq (7/4)|B_k|$. But then $(4/3)|B_k| \geq d_k \geq |B_n| \geq (7/4)|B_k|$. This is clearly a contradiction.

Thus, we can throw out all the intervals so that no three intervals intersect. Ordering the balls by their midpoints we can clearly separate these into two disjoint collections by putting every other ball in one collection and the remaining balls in another collection.

Hence we can cover A_n be two disjoint collections. Let these collections be called $\mathcal{F}_n^{(1)}$ and $\mathcal{F}_n^{(2)}$. Do this for all n. It is easy to see that we can patch these together, throwing away balls if necessary, to get $\mathcal{F}^{(1)}$ and $\mathcal{F}(2)$, disjoint collections, which cover all of A. ∎