Solution Set
Math 205a - Fall 2011
Problem Set 1

Problem 1 Let ϕ be a non-negative continuous function on $\mathbb{R}^n$ such that $\int \phi = 1$. Given $t > 0$ define $\phi_t(x) = t^{-n}\phi(x/t)$. Show that if $g \in C^\infty(\mathbb{R}^n)$ with compact support then

$$\phi_t(g) = \int_{\mathbb{R}^n} \phi_t(x)g(x)dx \to g(0)$$

First notice that $\int \phi_t = 1$ (by using a change of variables).

Fix $\epsilon > 0$. Since g is continuous and has compact support there exists $\delta > 0$ such that $|g(x) - g(0)| < \epsilon/2$ for all $|x| < \delta$. Also, let M be a number such that $|g(x)| < M$ for all x. Since, by the dominated convergence theorem,

$$\lim_{N\to\infty} \int_{[-N,N]} \phi = 1$$

then we can find $T > 0$ such that $0 < t < T$ implies that

$$\int_{[-\delta/t,\delta/t]} \phi = 1 - \frac{\epsilon}{4M}$$

Then we compute

$$\begin{aligned}
|\phi_t(g) - g(0)| &= |\int \phi_t(x)g(x)dx - \int \phi_t(x)g(0)dx| \\
&\leq \int_{[-\delta,\delta]} \phi_t(x)|g(x) - g(0)|dx + \int_{[-\delta,\delta]^c} \phi_t(x)|g(x) - g(0)|dx \\
&< \int_{[-\delta,\delta]} \phi_t(x)\frac{\epsilon}{2}dx + \int_{[-\delta/t,\delta/t]^c} 2M\phi(x)dx \\
&< \epsilon
\end{aligned}$$

It will be useful for the next problem to notice that one can easily slightly modify the proof so that it works for any bounded continuous function. ∎

Problem 2 Define

$$\Phi(t,x) = \begin{cases} (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Show that

(1) Show that $u(t,x) = \int \Phi(t, x - y)u_0(y)dy$ satisfies the heat equation
(2) That $u(t,x) \in C^\infty(\mathbb{R}^n \times (0,\infty))$
(3) That $\lim_{t\to 0} u(t,x) = u_0(x)$

We will do these for the special case of $n = 1$, but the general case is the same.

(1) Fix $t > 0$ and we'll prove that this $u(t,x)$ satisfies the heat equation on $[t/2, 3t/2]$. One can easily check that

$$\frac{\partial \Phi}{\partial t} = \Delta\Phi$$

Hence we need only show that we can bring the derivatives under the integral. This requires the justification of a theorem like the dominated convergence theorem. For now consider $h \in (0, t/2)$, but similar arguments will work when we take the limit from below. Now keep in mind a few facts. First, if $x > 0$ then $|e^{-x} - 1| \leq x$. Also, if $x > 0$ and $y > 0$ then $|\frac{\sqrt{y}}{\sqrt{y+x}} - 1| \leq x$. Hence we may bound

$$|\Phi(t+h,y) - \Phi(t,y)| \leq \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}} \left[\left|\frac{\sqrt{t}}{\sqrt{t+h}}(e^{-\frac{h}{t}\frac{1}{1+\frac{h}{t}}} - 1)\right| + \left|\frac{\sqrt{t}}{\sqrt{t+h}} - 1\right|\right] \leq \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}} \left[\frac{\sqrt{t}}{\sqrt{t+h}}\frac{h}{t} + h\right]$$

Hence it follows that $u_0(x-y)\frac{\Phi(t+h,y)-\Phi(t,y)}{h}$ is dominated by an integrable function. Hence we apply the dominated convergence theorem to get

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \lim_{t\to 0} \int u_0(x-y)\frac{\Phi(t+h,y) - \Phi(t,y)}{h} dy \\
&= \int u_0(x-y)\frac{\partial \Phi}{\partial t}(t,y)dy \\
&= \int u_0(y)\frac{\partial \Phi}{\partial t}(t,x-y)dy \\
&= \int u_0(y)\Delta_x\Phi(t,x-y)dy \\
&= \int u_0(y)\left(\lim_{h\to 0}\frac{\frac{\partial\Phi}{\partial x}(t,x+h-y) - \frac{\partial\Phi}{\partial x}(t,x-y)}{h}\right)dy
\end{aligned}$$

Hence we need to apply the dominated convergence theorem twice more (once for each derivative). We'll one case but the other works similarly. Notice that $\frac{\partial\Phi}{\partial x}(t,x) = p_t(x)\Phi(t,x)$ where $p_t(x)$ is some polynomial in x. Notice that there exists a polynomial g_t such that $|p_t(z+h) - p_t(z)| \leq g_t(z)h$ for all z and so that $|\Phi(t,z+h) - \Phi(t,z)| \leq h\Phi(t,z)g_t(z)$. Hence we get that

$$\begin{aligned}
\left|\frac{p_t(z+h)\Phi(t,z+h) - p_t(z)\Phi(t,z)}{h}\right| &\leq \frac{1}{h}[|\Phi(t,z+h)(p_t(z+h) - p_t(z))| + |(\Phi(t,z+h) - \Phi(t,z))p_t(z)|] \\
&\leq \Phi(t,z+h)g_t(z+h) + \Phi(t,z)g_t(z) \\
&\leq 2\Phi(t,z)G_t(z) + C\cdot 1_{[-t/2,t/2]}(z)
\end{aligned}$$

for some polynomial $G_t(z)$. Since this is integrable then we may pull the derivatives out of the integral using dominated convergence. Applying this twice gives us

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \int u_0(y)\Delta_x\Phi(t,x-y)dy \\
&= \frac{\partial}{\partial x}\int u_0(y)\frac{\partial\Phi}{\partial x}(t,x-y)dy \\
&= \frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Hence u satisfies the heat equation. ■

(2) This is simply iterating the arguments from part (1) to show that we may move the derivatives in and out of the integral by the dominated convergence theorem. ■

(3) Let $\phi(x) = (4\pi)^{-n/2}e^{-x^2/4} = \Phi(1,x)$. Then we may apply question 1 to see that

$$\lim_{t\to 0} u(t,x) = \lim \int \Phi(t,x-y)u_0(y)dy = \lim \int \frac{1}{\sqrt{t}^n}\phi(y/\sqrt{t})u_0(x-y)dy = u_0(x)$$

■

Problem 3 Construct a monotone function that is discontinuous on a dense set on $[0,1]$.

Let $\{r_1, r_2, \dots\} = \mathbb{Q}$. Define

$$f(x) = \sum_n 2^{-n} 1_{(-\infty,x)}(r_n)$$

This is clearly monotonic since it is the sum of increasing functions. Moreover it is discontinuous at each rational number since, for any n, if $x > r_n$, then it is easy to see that $f(x) - f(r_n) \geq 2^{-n}$. ■

Problem 4 Let E_k be a sequence of measurable sets such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty$$

Show that then almost all x lie in at most finitely many of the set E_k.

Now, we wish to show that the set of elements which occur in infinitely many E_k has measure zero. Notice that this set can be written as

$$E = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_k$$

Hence we get that

$$\mu(E) \leq \sum_{n=k}^{\infty} \mu(E_k)$$

Taking n to ∞ we get that $\mu(E) \leq 0$, as desired. ■

Problem 5 Construct a sequence of continuous function f_n such that $0 \leq f \leq 1$,

$$\lim_{n\to\infty} \int_0^1 f_n dx = 0$$

but that the sequence $f_n(x)$ converges for no $x \in [0,1]$.

We will do this using the identification $[0,1] = \mathbb{R}/\mathbb{Z}$. Let $0 \leq \phi \leq 1$ be a smooth function with support in $[-2,2]$ which is uniformly equal to 1 on $[-1,1]$. Let $x_n = \sum_{k=1}^{n} \frac{1}{k}$. Let $f_n(x) = \phi(n(x - x_n))$. Notice then that f_n is uniformly equal to 1 on the interval $[x_n - \frac{1/n}{,} x_n + \frac{1}{n}]$ and has support in $[x_n - \frac{2}{n}, x_n + \frac{2}{n}]$. Since x_n is unbounded, then it is easy to see that for any x and any N, there exists $n, m > N$ such that $f_n(x) = 1$ and $f_m(x) = 0$. Hence ϕ_n converges for no x. Also, it is easy to see that $|\int f_n| \leq \frac{4}{n}$. ■

Problem 6 Show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

One can easily check using induction that

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

Hence, by plugging in $-x^2$ we have that

$$1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} = \frac{1 + (-1)^n x^{2n+2}}{1 + x^2}$$

Let's attack both sides with $\int_0^1$. The left hand side becomes

$$1 - \frac{1}{3} + \cdots + (-1)^n \frac{1}{2n+1}$$

Taking the limit as n goes to ∞ we get

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Hence we just need to show that the limit of the integral of the right side is $\pi/4$. To see this notice that we can move the limit into the integral by dominated convergence (since the function is dominated by 2) to get

$$\frac{\pi}{4} = \tan^{-1}(1) = \int_0^1 \frac{1}{1+x^2} dx = \lim_{n\to\infty} \int_0^1 \frac{1+(-1)^n x^{2n+2}}{1+x^2} dx$$

■