Solution Set
Math 205a - Fall 2011
Problem Set 1

Problem 1 - 1 Show that f is well-defined, monotone, continuous and that it is constant on every interval contained in the complement of the Cantor set.

It is easy to show that if f x has two expansions, a_n and α_n then there exists N such that $a_k = \alpha_k$ for all $k \leq N$, $\alpha_{N+1} = 0$, $\alpha_k = 2$ for all $k > N+1$, $a_{N+1} = 1$ and $a_k = 0$ for all $k > N+1$. Let b_n and β_n be defined as in the statement of the problem for a_n and α_n respectively. Notice that $b_k = \beta_k$ for all $k \leq N$. Then

$$\sum_{n=1}^{N+1} \frac{b_n}{2^n} = \sum_{n=1}^{N} \frac{b_n}{2^n} + \frac{1}{2^{N+1}}$$

and

$$\sum_{n=1}^{\infty} \frac{\beta_n}{2^n} = \sum_{n=1}^{N} \frac{b_n}{2^n} + \sum_{n=N+2}^{\infty} \frac{1}{2^n}$$

Clearly these are equal and so f is well-defined.

Notice that $f(x) = \sum_{n=1}^{\infty} 2^{-n} R_n(x)$ where $R_n(x)$ is 1 if one of the intervals removed during the nth step of the creation of the Cantor set is contained competely between 0 and x and is 0 otherwise. Then monotonicity is clear, since $R_n(x) \leq R_n(y)$ for all n if $x < y$. Moreover, continuity is clear since if $|x - y| < 3^{-N}$ then for all $n < N$ we have that $R_n(x) = R_n(y)$ and hence $|f(y) - f(x)| \leq \sum_{n=N}^{\infty} \frac{1}{2^n} \leq 2^{-N+1}$. That f is constant on every interval contained in the complement of the Cantor set is clear since for x, y in such an interval, $R_n(x) = R_n(y)$ for every n. ∎

Problem 2 - 2 Show that g is a homeomorphism of $[0,1]$ onto $[0,2]$ and that $m[g(C)] = 1$. Show that there exists a measurable set A so that $p^{-1}(A)$ is not measurable. Show that there is a measurable set that is not a Borel set.

Notice that g is strictly increasing and hence one-to-one. Since it is continuous and since $g(0) = 0$ and $g(1) = 1$ then we have that $g([0,1]) = [0,2]$. Notice that $p^{-1}(a,b) = (g(a), g(b))$. Hence p must be continuous. Hence g is a homeomorphism.

We claim that for any set $A \subset [0,2]$ such that $m(A) > 0$, there exists $E \subset A$ which is not measurable. In the same way as in class, let P be a set which contains exactly one representative of every equivalence class of numbers with the relation that x y iff $x = y + q$ for some $q \in \mathbb{Q}$. Let $r_1, r_2, \ldots$ be the rationals in $[0,2]$. Let $P_i = P + r_i$ (where we do addition mod 2). Let $E_i = A \cap P_i$. If any E_i is not measurable then we're done. So assume that E_i is measurable for all i. Since P_i is not measurable, if $m(E_i) > 0$ then we have that $\infty \sum_j m(E_i + r_j) = m(\cup_j E_i + r_j) \leq m([0,2])$. This is clearly a contradiction. Hence $m(E_i) = 0$. On the other hand, we have that $m(A) = m(\cup_i E_i) = \sum m(E_i) = 0$. This is clearly a contradiction.

Take E be to such a non-measurable subset of $g(C)$. Let $B = g^{-1}(E)$. Then since $B \subset C$ we have that $m(B) = 0$ and hence B is measurable. However, $p^{-1}(B) = E$ is not measurable. Note that this also gives us a measurable set that is not Borel, since if B were Borel then $p^{-1}(B)$ would be measurable. ∎

Problem 3 - 3

(1) lower semicontinuous if and only if $\{x : f(x) > \lambda\}$ open for all λ
(2) If f, g lower semicontinuous then $f + g$ and $\max\{f, g\}$ are.
(3) If f_n are lsc, so its $f(x) = \sup_n f_n(x)$
(4) A function f is lsc if and only if there is a monotone increasing sequence ϕ_n of continuous functions such that $f(x) = \lim \phi_n(x)$.

(i) First assume that f is lower semicontinuous. Fix λ. Let $A_{\lambda,f} = \{x : f(x) > \lambda\}$. Fix $x \in A_{x,\lambda}$. If f is lower semicontinuous then for each $\epsilon < f(x)-\lambda$ there exists $\delta > 0$ such that if $\eta < \delta$ then $\inf_{|x-y|<\eta} f(y) \geq f(x)-\epsilon$. Then if $|y - x| < \delta$, we have that

$$f(y) \geq f(x) - \epsilon > \lambda$$

Hence $B(x, \delta) \subset A_{f,\lambda}$.

Now suppose that $A_{f,\lambda}$ is open for all $\lambda > 0$. Fix x. Fix $\epsilon > 0$ and let $\lambda = f(x) - \epsilon$. Then since $A_{f,\lambda}$ is open, there is δ such that for all $|x - y| < \delta$ we have that $f(y) > f(x) - \epsilon$. Hence it follows that $\liminf f(y) \geq f(x) - \epsilon$. Since this is true for all ϵ, then $\liminf f(y) \geq f(x)$. ■

(ii) Let $h = \max\{f, g\}$. Then $A_{h,\lambda} = A_{f,\lambda} \cup A_{g,\lambda}$ is open, and hence h is lsc. Let $k(x) = f(x) + g(x)$. Then $A_{k,\lambda} = \cup_{r\in\mathbb{Q}} A_{f,r} \cap A_{g,\lambda-r}$ is open, and hence k is lsc. ■

(iii) Let $h(x) = \sup f_n(x)$. Then $A_{h,\lambda} = \cup_n A_{f_n,\lambda}$ is open and hence h is lsc. ■

(iv) One direction follows from the fact that $\lim \phi_n = \sup \phi_n$ and part (iii).

For the other direction. Define $g_n(x) = \inf_{z\in[a-2^{-(n+1)},a+2^{-(n+1)})} f(z)$, where $a = k/2^n$ for some k and such that $x \in [a - 2^{-(n+1)}, a + 2^{-(n+1)})$. One can check using the compactness of the closure of such an interval the infimum is larger than $-\infty$. Now we can easily get functions f_n which are continuous, equal to g_n on the set $[a - 2^{-(n+2)}, a + 2^{-(n+2)}]$, $f_n \leq g_n$, and which are linear on the intervals in which $f_n \neq g_n$. Let $\phi_n(x) = \max\{f_1(x), \ldots, f_n(x)\}$. This is clearly monotone and continuous. Also, we have that for all n, $\phi_n \leq f_n \leq g_n \leq f$. Now fix $x \in \mathbb{R}$ and let $a = \lim_n \phi_n(x)$, and suppose that $a < f(x)$. Let $\lambda < f(x) - a$ and since $A_{f,\lambda}$ is open, we have some $\delta > 0$ such that if $|x - y| < \delta$ then we have that $f(y) \geq \lambda$. Hence for n large enough, we have by the construction of g_n and f_n that $f_n(x) \geq \lambda$. Then by the construction of ϕ_n we have that $\phi_n(x) > f(x) - a$. This is a contradiction. ■

Problem 4 - 4

(1) The set of points of continuity of f is a $\mathcal{G}_\delta$.
(2) $\mathbb{Q}$ is not a $\mathcal{G}_\delta$

(i) Let $V_n = \{x :$ there exists $\delta > 0$ such that $|z - x|, |y - z| < \delta$ implies that $|f(z) - f(y)| < 1/n\}$. One can quickly check that this is open and that the set of points of continuity is equal to $\cap_n V_n$. ■

(ii) Suppose that $\mathbb{Q} = \cap V_n$ where V_n is countable. Let $r_1, r_2, \ldots$ be the rationals. Let $U_n = \mathbb{R} \setminus \{r_n\}$. Then U_n and V_n are dense and open. Hence by the Baire Category theorem we should have that $(\cap U_n)\cap(\cap V_n) \neq \emptyset$. However, this set is clearly equal to $(\mathbb{R} \setminus \mathbb{Q}) \cap \mathbb{Q} = \emptyset$. ■

Problem 5 - 5 Prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure on $\mathcal{M}$.

Clearly $\emptyset$ is in $\mathcal{M}$. Also, if $A \in \mathcal{M}$ then A^c is clearly also in $\mathcal{M}$. Now suppose that $A_1, \ldots, A_n, \ldots$ are elements of $\mathcal{M}$. If all $A_1, \ldots, A_n, \ldots$ are countable then $\cup A_n$ is countable and so is in $\mathcal{M}$. Otherwise, WLOG A_1^c is countable. Since $(\cup_n A_n)^c \subset A_1^c$, then it is countable. Hence $\cup_n A_n$ is in $\mathcal{M}$ and this is in fact a σ algebra.

First notice that μ is clearly non-negative and that $\mu(\emptyset) = 0$. Let $A_1, \ldots, A_n, \ldots$ be a sequence of measurable sets. If A_n is countable for all n then $\mu(\cup A_n) = 0 \leq \sum 0 = \sum \mu(A_n)$. If WLOG A_1 has countable complement, then $\mu(\cup A_n) = 1 = \mu(A_1) \leq \sum \mu(A_n)$. Hence μ is a measure. ■

Problem 6 - 6 Construct a Borel set E such that for every interval I, $0 < m(E \cap I) < m(I)$.

Fix $\alpha \in (0, 1)$. Notice that if create a Cantor-type set C_α by removing 2^{n-1} intervals of length δ^n for $\delta = \frac{\alpha}{1+2\alpha}$ at each step (in the same way we removed interval for the Cantor set), we will be left with a set of measure $\sum \delta^n 2^{n-1} \delta^n = \frac{\delta}{1-2\delta} = \alpha$. It is easy to check, in the same way as with the usual Cantor set, that C_α is closed and nowhere dense.

Now let I_n be the collection of all intervals with rational endpoints. Define $A_1, A_2, \ldots$, $B_1, B_2, \ldots$, and $E_1, E_2, \ldots$ inductively as follows. Let

$$E_n = \left(\bigcup_{i=1}^{n-1} A_i \right) \cup \left(\bigcup_{i=1}^{n-1} B_i \right)$$

Then since, inductively, E_n is nowhere dense, as A_i, B_i are nowhere dense, we have that $I_n \setminus E_n$ contains two non-empty intervals, $J_1^{(n)}, J_2^{(n)}$. Let A_n be a translation and dilation of $C_{1/2}$ which is a subset of $J_1^{(n)}$ and let B_n be a translation and dilation of $C_{1/2}$ which is a subset of $J_2^{(n)}$. Hence we have that $I_n \cap A_n$ and $I_n \cap B_n$ have positive measure.

Let $A = \cup A_n$ and $B = \cup B_n$. Notice that $A \cap B = \emptyset$ by construction. Let I be a non-empty interval and then there exists k such that $I_k \subset I$. Since $A_k \cap I_k \subset A \cap I$ then $m(A \cap I) > 0$. Similarly for $m(B \cap I) > 0$. Since $A^c \cap I \supset B \cap I$, then it follows that

$$0 < m(A \cap I) < m(I)$$

■

Problem 7 - 7 Show that there is a support of every Borel measure. Show that every compact set is the support of some Borel measure.

Let $\mathcal{F} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{Q}, \mu(a, b) = 0\}$. If $\mathcal{F} = \emptyset$ then all open sets must have measure 0 and we can take $K = [0, 1]$. If not, then let $V = \cup_{I \in cF} I$. This is open and has measure zero, since it is a countable union of measure zero sets. Let $K = V^c$. We claim that K is the set we are looking for. Notice that $\mu(K) = 1$. Suppose that $H \subset K$ and that $x \in K \setminus H$. Then every interval (a, b), where $a, b \in \mathbb{Q}$, containing x must have positive measure. Fix such an a, b such that $(a, b) \cap H = \emptyset$. Then $\mu(H) \leq 1 - \mu(a, b) < 1$.

Now suppose that K is a non-empty compact set. It is a fact, easy to verify, that there exists a countable dense subset $\{k_1, k_2, \ldots\} \subset K$. This set is either finite or countably infinite. If it is finite, let $k_l = k_N$ for all $l \geq N$, where $k_1, \ldots, k_N$ is the full set. Then let $\mu(S) = \sum_{n=1}^{\infty} 2^{-n} \delta_{k_n}(S)$. It is easy to check that K is in fact the support of μ. ■

Problem 8 - 8 Is $f(x) = \mu(x + V)$ upper or lower semicontinuous?

f is not upper semicontinuous. For instance, consider $\mu = \delta_0$. To see that f is not lower semicontinuous, suppose it is not at some point x. WLOG consider $x = 0$. Then there exists $x_n \to 0$ such that $\mu(V+x_n)+\epsilon < \mu(V)$ for some $\epsilon > 0$. We can assume that $|x_n| < 1/n$ for all n, otherwise we can take a subsequence. Let $V_n = \{x \in V : d(x,V) > 1/n\}$. Then notice that $x_n + V_n \subset x_{n+1} + V_{n+1} \subset V$ for all n and that $\cup_n V_n + x_n = V$. Then

$$\mu(V) = \mu(\cup V_n + x_n) = \lim \mu(V_n + x_n) \leq \liminf \mu(V + x_n) < \mu(V) - \epsilon$$

This is clearly a contradiction.

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