Homework # 6.

1. The Poisson kernel $P(z) : D_1 \times \partial D_1 \to \mathbb{R}$ (here $D_1 = \{|z| \le 1\} \subset \mathbb{C}$ is the unit disk) is defined as

$$P(z,\zeta) = \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1-|z|^2}{|\zeta - z|^2},\ |z| < 1,\ |\zeta| = 1.$$

If $D(w,\rho) = \{z \in \mathbb{C} :\ |z-w| < \rho\}$ is an open disk, and $\phi :\ \partial D(w,\rho) \to \mathbb{R}$ is an $L^1(\partial D)$ function, then its Poisson integral is

$$P_D(\phi)(z) = \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{z-w}{\rho}, e^{i\theta}\right)\phi(w+\rho e^{i\theta})d\theta,\ z \in D(w,\rho).$$

More explicitly,

$$P_D\phi\left(w+re^{it}\right) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}\phi\left(w+\rho e^{i\theta}\right)d\theta.$$

Show that the function $P_D\phi$ is harmonic in $D(w,\rho)$ and that if ϕ is continuous at $\zeta_0 \in \partial D$, then $\lim_{z\zeta_0} P_D\phi(z) = \phi(\zeta_0)$. Hint: show that $P > 0$, and $\frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})d\theta = 1$, and $\sup_{|\zeta-\zeta_0|\ge\delta} P(z,\zeta) \to 0$ as $z \to \zeta_0$, for $|\zeta_0| = 1$ and $\delta > 0$.

2. Show that among sets of the same diameter the ball has the largest volume. Hint: use symmetrization.

3. Suppose that $\{g_n\}$ is a sequence of positive continuous functions on $[0,1]$, μ is a positive Borel measure on $[0,1]$ and that (i) $\lim_{n\to\infty} g_n(x) = 0$ a.e., (ii) $\int_0^1 g_n dx = 1$ for all n and (iii) $\lim_{n\to\infty}\int_0^1 f g_n dx = \int_0^1 f d\mu$ for every continuous function $f \in C[0,1]$. Does it follow that μ is mutually singular with respect to the Lebesgue measure?

4. Let $X(n;m)$, $n \in \mathbb{N}$, $m \in \mathbb{Z}^2$ be the standard random walk on the 2-dimensional integer lattice which starts at time $n = 0$ at the point $m \in \mathbb{Z}^2$. This means that the particle starts at $X(0;m) = m$ and at each time step n the particle can jump from the current position $X(n;m)$ to any of the 4 points of the form $X(n;m) \pm e_j$, where e_j is one of the two standard basis vectors in $\mathbb{R}^2$ with equal probabilities, all equal to 1/4. In other words, $P(\{X(n+1;m) = y\}) = 1/4$ for all nearest neighbors y of $X(n;m)$.

(i) Let γ be a non-intersecting path on the lattice which surrounds the starting point m and let $f(y)$ be a bounded function on γ. Let also $\tau \in \mathbb{N}$ be the first time when the path $X(n;m)$ hits γ. Define the function $g(m) = E\{f(X(\tau,m)\}$, the expected value of the function f evaluated at the point where $X(n,m)$ exits the domain bounded by γ. Show that $g(m)$ satisfies the discrete Laplace equation

$$Dg(m) := g(m) - \frac{1}{4}\sum_y g(y) = 0,$$

where the sum is taken over the 4 neighbors of m. This equation is supplemented by the boundary condition $g(m) = f(m)$ for $m \in \gamma$. Interpret $g(m)$ when $f(y)$ is the characteristic function of a subset of γ.

(ii) Let $\phi(m) = E\{\tau(m)\}$ where $\tau(m)$ is the aforementioned exit time of the random walk $X(n;m)$ from the domain bounded by γ. Show that $\tau(m)$ satisfies the discrete Poisson equation

$$Dg(m) = 1,$$

supplemented by the boundary condition $g(m) = 0$ for $m \in \gamma$.

(iii) Let also $f(x)$ be a bounded function on the lattice $\mathbb{Z}^d$ and define $g(n,m) = E\{f(X(n;m))\}$, that is, the expected value of the function f evaluated at the point where the process X is at time n if it started at the time $n = 0$ at position m. Show that $g(n,m)$ satisfies the discrete heat equation with the initial condition $g(0,m) = f(m)$.

Now, formally assume that the time steps are of the order δt and the lattice distance is δx with $\delta t = (\delta x)^2$. What are the limits of these discrete equations?

5. (i) Let $g_n = \chi_{[-n,n]}(x)$, compute $h_n = g_n \star g_1$ and show that h_n is a Fourier transform of a multiple of the function

$$f_n(x) = \frac{\sin x \sin(nx)}{x^2}.$$

(ii) Use (i) to show that the Fourier transform maps L^1 into a proper subset of $C_0(\mathbb{R})$ and not onto $C_0(\mathbb{R})$.
(iii) Show that the image of the Fourier transform is dense in $C_0(\mathbb{R})$.

6. (i) Let $f \in C(\mathbb{S}^1)$ have a modulus of continuity $\omega(\delta) = \sup_{|x-y|\le\delta} |f(x) - f(y)|$. Show that $|\hat{f}(n)| \le C\omega(1/2n)$.
(ii) Assume that f is absolutely continuous, show that $\hat{f}(n) = o(1/n)$ as $n \to +\infty$.
(iii) Show that $f \in L^1(\mathbb{S}^1)$ is equal to an analytic function a.e. on $\mathbb{S}^1$ if and only if there exist $c > 0$ and $A > 0$ so that $|\hat{f}(n)| \le Ae^{-cn}$.

7. Let $f(x) = x$ for $0 \le x < 1$ and extend it periodically to $\mathbb{R}$. Consider the partial sums $S_N f(x)$. Given a sequence $x_N \to 0$, what are the possible limits of $(S_N f)(x_N)$?