Homework # 5.

1. Let $B(x,\delta)$ be a ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$ and define

$$\tilde{M}f(x) = \sup_{\delta>0} \frac{1}{m(B(y,\delta))} \int_{B(y,\delta)} |f(y)|dy.$$

with the supremum taken over all balls $B(y,\delta)$ such that $x \in B(y,\delta)$.
(i) Use the covering lemmas to show that there exists a constant $c > 0$ that depends only on dimension n so that (m is the n-dimensional Lebesgue measure)

$$m\{x: \ \tilde{M}f(x) > \alpha\}) \le \frac{c}{\alpha} \int_{\mathbb{R}^n} |f(y)|dy.$$

Now, show that if $f \in L^p(\mathbb{R}^n)$, with $p > 1$, then $\tilde{M}f \in L^p(\mathbb{R}^n)$. Hint: introduce $f_1(x) = f(x)$ if $|f(x)| > \alpha/2$ and $f_1(x) = 0$ if $|f(x)| \le \alpha/2$ and show that $\{\tilde{M}(f) > \alpha\} \subset \{\tilde{M}(f_1) > \alpha/2\}$ so that

$$m(\{x: \ \tilde{M}f(x) > \alpha\}) \le \frac{c}{\alpha} \int_{|f|\ge\alpha/2} |f(y)|dy.$$

Then use the relation

$$\int (\tilde{M}f)^p dy = p \int_0^\infty m(\tilde{M}f > \alpha)\alpha^{p-1} d\alpha$$

to finish the proof. This also proves that $\tilde{M}f$ is finite a.e.
(ii) Show that if $f \in L^1$ and $f \neq 0$ identically then there exist $C, R > 0$ so that $\tilde{M}f(x) \ge C|x|^{-n}$ (here n is the space dimension) for all x with $|x| \ge R$. Hence $m(\{x: \ \tilde{M}(x) > \alpha\}) \ge C'/\alpha$.

2. Set

$$Mf(x) = \sup_{\delta>0} \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} |f(y)|dy.$$

with the supremum taken over all balls $B(x,\delta)$.
(i) Show that $M(x) \le \tilde{M}(x) \le 2^n M(x)$.
(ii) Show that $|f(x)| \le Mf(x)$ at every Lebesgue point of f if $f \in L^1(\mathbb{R}^n)$.

3. A measurable function f is said to be in weak L^1 denoted by L^1_w if $m(\lambda) = \lambda \times |\{x: \ |f(x)| > \lambda\}|$ is a bounded function of $\lambda \ge 0$. Show that L^1_w contains L^1 but is larger than L^1.

4. Let $f(x) \in C(\mathbb{R})$, $f(x) > 0$ for $0 < x < 1$ and $f(x) = 0$ otherwise. Show that the function $h_c(x) = \sup_n\{n^c f(nx)\}$ is (i) in $L^1(\mathbb{R})$ if $c \in (0,1)$, (ii) is in $L^1_w(\mathbb{R})$ but not in $L^1(\mathbb{R})$ if $c = 1$, (ii) not in L^1_w if $c > 1$.

5. Suppose X consists of two points: a and b. Define $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(X) = \infty$ and $\mu(\emptyset) = 0$. Is it true for this μ that the dual of $L^1(\mu)$ is $L^\infty(\mu)$?

6. Let $a_n(x) = a(nx)$ where $a(x)$ is a smooth periodic function with period one: $a(x+1) = a(x)$. Let $f(x)$ be a smooth funcion on $[0,1]$. Find the limit of

$$\int_0^1 f(x)a_n(x)dx.$$

How much can you relax the smoothness assumptions on $a(x)$ and $f(x)$?

7. Extra for people with free time: Let

$$\phi_0(t) = \begin{cases} 1, & x \in [0,1] \\ -1, & x \in [1,2) \end{cases},$$

extend it periodically to all of $\mathbb{R}$, and define $\phi_n(t) = \phi_0(2^n t)$, $n \in \mathbb{N}$. Assume that $\sum |c_n|^2 < \infty$ and show that the series

$$\sum_{n=1}^{\infty} c_n \phi_n(t)$$

converges for almost every t. Yes, you can find this in a book.