Homework # 4.

1. Let $g \in L^q[0,1]$ and define a mapping $G : L^p[0,1] \to \mathbb{R}$ by $G(f) = \int_0^1 fg dx$. Show that G is a bounded linear functional on $L^p[0,1]$, that is, G is linear and there exists a constant $C > 0$ so that $|G(f)| \le C\|f\|_{L^p}$ for all $f \in L^p$. Here $1/p + 1/q = 1$.

2. Let f_n be a sequence of functions in L^p, $1 < p < \infty$ which converge almost everywhere pointwise to a function $f \in L^p$. Suppose that $\|f_n\|_p \le M$ and show that for all $g \in L^q$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ we have $\int f_n g \to \int fg$. Is this result true with $p = 1$ and $q = \infty$? How about $p = \infty$ and $q = 1$?

3. Let $f(t) \in L^p(\Omega)$, where $\Omega \in \mathbb{R}^n$. We say that

$$\mu(t) = m\{x \in \Omega : \ |f(x)| > t\}$$

is the distribution function of $|f|$. Show that (this is a version of the Chebyshev inequality)

$$\mu(t) \le \frac{1}{t^p}\|f\|^p_{L^p(\Omega)}$$

and

$$\|f\|^p_{L^p(\Omega)} = p\int_0^\infty t^{p-1}\mu(t)dt.$$

More generally, show that we have for any differentiable increasing function $\phi(s)$ such that $\phi(0) = 0$:

$$\int_\Omega \phi(|f(x)|)dx = \int_0^\infty \phi'(\lambda)\mu(\lambda)d\lambda.$$

5. Given two functions f and g we define their convolution as $f \star g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$. Show that if $f \in L^p$, $1 \le p < \infty$, $\phi \ge 0$, $\phi \in L^1$ with $\int \phi = 1$ and $\phi_t = t^{-m}\phi(x/t)$, then

$$\lim_{t\to 0}\|\phi_t \star f - f\|_p = 0.$$

6. Let μ be a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(X; d\mu)$ and let

$$\alpha_n = \int_X |f|^n d\mu, \ n \in \mathbb{N}.$$

Prove that

$$\lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

7. Show that the Fubini theorem holds for non-negative functions without a priori knowledge that the function is summable in $X \times Y$. Let μ and ν be σ-finite measures on X and Y, respectively. Let $f \ge 0$ be a measurable function on $X \times Y$. Show that then (i) for almost all x the function $f(x,y)$ is ν-measurable (as a function of y with x being a parameter) and for almost all y the function $f(x,y)$ is μ-measurable (as a function of x with y being a parameter); (ii) $\displaystyle\int f(x,y)d\mu(x)$ is a measurable function of y and $\displaystyle\int f(x,y)d\nu(y)$ is a measurable function of x; (iii)

$\int_{X\times Y} f d(\mu \times \nu) = \int_X \left[\int_Y f d\nu\right] d\mu = \int_Y \left[\int_X f d\mu\right] d\nu$. (iv) Show that the assumption that f is non-negative may not be removed: take $X = Y = \mathbb{N}$, let $\mu = \nu$ be the counting measure ($\mu(E)$ is the number of elements of E is E is finite, otherwise $\mu(E) = \infty$) and let

$$f(n, m) = \begin{cases} 1, & n = m \\ -1, & n = m + 1. \\ 0, & \text{otherwise} \end{cases}$$

Show that $\int |f| d(\mu \times \nu) = \infty$, while $\int (\int f d\mu) d\nu$ and $\int (f d\nu) d\mu$ both exist and are not equal.

8. Prove the Besikovitch theorem in the one-dimensional case by hand. What is the smallest number of disjoint collections that you will need?