

Homework # 2.

1. Let ϕ be a non-negative continuous function on $\mathbb{R}^n$ such that $\int \phi = 1$. Given $t > 0$ define $\phi_t(x) = t^{-n}\phi(x/t)$. Show that if $g \in C^\infty(\mathbb{R}^n)$ with compact support then

$$\phi_t(g) = \int_{\mathbb{R}^n} \phi_t(x)g(x)dx \rightarrow g(0).$$

Because of that ϕ_t is called an approximation of identity.

2. Let $u_0(x)$ be a continuous bounded function on $\mathbb{R}^n$. The heat kernel is defined by

$$\Phi(t, x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0. \\ 0, & t < 0 \end{cases}.$$

Show that (i) the function

$$u(t, x) = \int_{\mathbb{R}^n} \Phi(t, x - y)u_0(y)dy$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = \Delta u := \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2},$$

(ii) that $u(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ – this is the regularizing effect of the heat equation: $u(x, t)$ is smooth even if $u_0(x)$ is not, and

(iii) that $\lim_{t \rightarrow 0} u(t, x) = u_0(x)$ for all $x \in \mathbb{R}^n$.

3. Construct a monotone function that is discontinuous on a dense set on $[0, 1]$.

4. Let E_k be a sequence of measurable sets such that

$$\sum_{k=1}^{\infty} \mu(E_k) < +\infty.$$

Show that then almost all x lie in at most finitely many of the sets E_k .

5. Construct a sequence of continuous functions f_n such that $0 \leq f \leq 1$,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n dx = 0,$$

but the sequence $f_n(x)$ converges for no $x \in [0, 1]$.

6. A compact set K is the support of a measure μ with $\mu(\mathbb{R}) = 1$ if $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K . Show that every compact set is the support of a Borel measure.

7. Construct a measure on the line such that its support is the standard Cantor set on $[0, 1]$.

8. Show that (no, this has nothing to do with anything in case you wonder)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$