Homework # 1.

The ternary expansion of a real number $r \in [0,1]$ is its representation as $r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n = 0, 1, 2$. The sequence $\{a_n\}$ is called the ternary expansion of r. Such an expansion is unique (up to a tail of two's) for a given $r \in [0,1]$. The ternary Cantor set C consists of all real numbers in $[0,1]$ such that $a_n \neq 1$ for all $n \geq 1$ (if r has two ternary expansions, we put it into C if one of the expansions contains no ones). Recall (or show if you do not remember) that the Cantor set is closed, uncountable and has the Lebesgue measure equal zero.

1. Let $x \in [0,1]$ have a ternary expansion $\{a_n\}$. Let $N = \infty$ if none of a_n are 1, otherwise let N be the smallest value of n such that $a_n = 1$. Let $b_n = a_n/2$ for $n < N$ and $b_N = 1$. Show that the sum $f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}$ is independent of the ternary expansion if x has two expansions. Show that $f(x)$ is a monotone continuous function on $[0,1]$ that is constant on each interval contained in the complement of the Cantor set.

2. Let $f(x)$ be the Cantor function, and let $g(x) = f(x) + x$. Show that g is a homeomorphism (g^{-1} is continuous) of $[0,1]$ onto $[0,2]$, that $m[g(C)] = 1$ (C is the Cantor set). Let $p(x) = g^{-1}$. Show that there exists a measurable set A so that $p^{-1}(A)$ is not measurable. Show that there is a measurable set that is not a Borel set.

3. A function f is *lower semi-continuous* (lsc) at a point y if $f(y) \leq \liminf_{x \to y} f(x)$. Show that (i) a function f is lower semi-continuous if and only if the set $\{f(x) > \lambda\}$ is open for all $\lambda \in \mathbb{R}$, (ii) if f and g are lsc at y then so are $\max(f, g)$ and $f + g$, (iii) if f_n are lsc, so is $f(x) = \sup_n f_n(x)$, (iv) a function f is lsc if and only if there is a monotone increasing sequence $\{\phi_n\}$ of continuous functions so that $f(x) = \lim_{n \to \infty} \phi_n(x)$.

4. Recall that a set is of the class $\mathcal{G}_\delta$ if and only if it is an intersection of a countable collection of open sets. (i) Show that the set of points of continuity of a function f is $\mathcal{G}_\delta$. (ii) Show that the set of all rational numbers is not a $\mathcal{G}_\delta$ set.

5. Let X be an uncountable set, let $\mathcal{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable. Define $\mu(E) = 0$ in the first case and $\mu(E) = 1$ in the second. Prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure on $\mathcal{M}$ (that is, all sets in $\mathcal{M}$ are measurable).

6. Construct a Borel set $E \subset \mathbb{R}$ so that $0 < m(E \cap I) < m(I)$ for every nonempty interval I.

7. Let μ be a Borel measure on $[0,1]$ with $\mu([0,1]) = 1$. Show that there exists a compact set $K \subseteq [0,1]$ so that $\mu(K) = 1$ but $\mu(H) < 1$ for any proper compact subset H of K. K is called the support of μ. Show that every compact subset of $[0,1]$ is the support of some Borel measure.

8. Let V be an open subset $\mathbb{R}^n$ and μ be a finite Radon measure. Define $f(x) = \mu(V + x)$, does the function $f(x)$ have to be continuous, lower semicontinuous or upper semicontinuous?