Final Exam, Math 205, Fall 2011

1. Let A be a compact subset of $L^1[0,1]$. Show that A satisfies the following condition: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $f \in A$ and any measurable set B with $m(B) < \delta$ we have $\int_B |f| < \varepsilon$.

2. Let $H^s(\mathbb{R}^n)$ be the set of all Schwartz distributions f in $\mathcal{S}'(\mathbb{R}^n)$ whose Fourier transform is a function $\hat{f}$ such that

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2} |\hat{f}(\xi)|^2 d\xi < +\infty.$$

(i) For which $s \in \mathbb{R}$ does the Dirac delta function $\delta(x)$ lie in $H^s(\mathbb{R}^n)$?
(ii) Show that there exists $S(n)$ such that every element of $H^s(\mathbb{R}^n)$ with $s > S(n)$ is a bounded function.

3. Let $f \in L^1(\mathbb{T})$ be a periodic function on the interval $[0,1]$, its Fourier transform is defined as

$$\hat{f}_n = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Set

$$S_N f(x) = \sum_{n=-N}^{N} e^{2\pi i n x} \hat{f}_n,$$

and

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n f(x).$$

Assume that $|f(x+y) - f(x)| \leq C|y|^\alpha$ with $0 < \alpha < 1$ for all $0 \leq x, y \leq 1$. Show that then there exists a constant K (which depends on f but not on n or x) so that

$$|\sigma_n f(x) - f(x)| \leq \frac{K}{n^\alpha}$$

for all x. Hint: split the integration over $|x| < \delta$ and $\delta \leq |x| \leq 1$, and optimize over δ.

4. Fix two sequences of integers $m_k \to +\infty$ and $N_k \to +\infty$ and set

$$B_k(x) = \sum_{n=-m_k}^{m_k} \frac{e^{2\pi i n x}}{in}$$

and

$$f(x) = \sum_{k=1}^{\infty} \frac{B_k(x)}{k^2} e^{iN_k x}.$$

(i) Show that there exists a constant $M > 0$ so that $|B_k(x)| \leq M$ for all k and that the function $f(x)$ is continuous on $[0,1]$.
(ii) Assume that $N_{k+1} - N_k > m_k + m_{k+1}$ and show that then the sequence $S_{N_k} f(0)$ of the Fourier partial sums diverges.

5. (i) Let $\phi \in C^k(S^1)$ and $\psi \in C^m(S^1)$. Show that the convolution $\phi \star \psi$ is in $C^{k+m}(S^1)$. The convolution on the circle is defined as

$$\phi \star \psi(x) = \int_{S^1} \phi(x-y)\psi(y) dy.$$

(ii) Give an example of a function $\psi \in C(S^1)$ such that $\psi \star \psi \star \ldots \star \psi$ (k times) is not differentiable for any k.

6. The Hausdorff-Young inequality claims that for $1 \leq p \leq 2$ and $1/p + 1/q = 1$ we have

$$\|\hat{f}\|_{L^q(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}. \tag{1}$$

Here $\hat{f}$ is the Fourier transform of f. Show the converse: that is, if (1) holds for all $f \in L^p(\mathbb{R})$, then $1/p + 1/q = 1$ and $1 \leq p \leq 2$.

7. (a) Let $B \in C^1([0,1])$ be a subspace of dimension $N+1$. Show that there exists $f \in B$ such that $\sup_{0 \leq x \leq 1} |f(x)| = 1$ and $\sup_{0 \leq x \leq 1} |f'(x)| \geq 2N$. Hint: there has to be a function $g \in B$ which vanishes sufficiently often.

(b) Prove that a subspace $B \subset C^1([0,1])$ which is closed under uniform convergence is finite dimensional. Hint: Show that $\|f'\|_\infty \leq K\|f\|_\infty$ for some constant K and all $f \in B$.

8. Let S be a subspace of $C[0,1]$. Suppose S is closed as a subspace of $L^2[0,1]$. Prove: (a) S is a closed subspace of $C[0,1]$. (b) For $f \in S$, $\|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2$. (c) For every $y \in [0,1]$, there is a $K_y \in L^2[0,1]$ such that

$$f(y) = \int K_y(x) f(x) dx$$

for every $f \in S$.