MATH 172, SPRING 2010
SOLUTION TO MIDTERM

(PROBLEM 0)
We define the Lebesgue integral for non-negative functions by:

$$\int_E f = \sup\left\{\int_E h : h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E\right\}$$

To show $\int_E f = 0$, it suffices to show $\int_E h = 0$ for all bounded, measurable functions h of finite support and $h \geq 0$ on E.
Let h be such a function, its Lebesgue integral over E is defined to be

$$\int_E h = \sup\left\{\int_E \phi : \phi \text{ simple and } \phi \leq f \text{ on } E\right\}$$

Write $\phi = \sum_{i=1}^{N} c_i \chi_{E_i}$ where each $E_i \subset E$ and $c_i \in \mathbb{R}$, then

$$\int_E \phi = \sum_{i=1}^{N} c_i m(E_i) = \sum_{i=1}^{N} c_i \cdot 0 = 0$$

Therefore, $\int_E h = 0$ and hence $\int_E f = 0$.

(PROBLEM 1)
(i) For M_n, we will prove $\{x : M_n(x) \leq c\}$ is a measurable set for any $c \in \mathbb{R}$. Note that

$$M_n(x) = \sup_{k>n} f_k(x) \leq c \Leftrightarrow f_k(x) \leq c \text{ for all } k > n.$$

so we have $\{x : M_n(x) \leq c\} = \cap_{k>n}\{x : f_k(x) \leq c\}$. Since each f_k is a measurable function, each $\{x : f_k(x) \leq c\}$ is a measurable set. Therefore $\{x : M_n(x) \leq c\}$ is a measurable set for every $c \in \mathbb{R}$, and so M_n is a measurable function.
Similarly, we have $\{x : m_n(x) \geq c\} = \cap_{k>n}\{x : f_k(x) \geq c\}$. Therefore, m_n is also a measurable function.
(ii) $f_k(x)$ converges as $k \to \infty$ if and only if $\liminf f_n(x) = \limsup f_n(x) \neq \pm\infty$. Note that

$$\liminf f_n(x) = \sup_{n>1} \inf_{k>n} f_k(x)$$
$$\limsup f_n(x) = \inf_{n>1} \sup_{k>n} f_k(x)$$

By (i), $\inf_{k>n} f_k(x)$ is a measurable function for all $n > 1$, and hence $\liminf f_n(x) = \sup_{n>1}(\inf_{k>n} f_k(x))$ is also a measurable function. By the same reason, $\limsup f_n(x)$ is also measurable. This implies $\limsup f_n - \liminf f_n$ is a measurable function.

$$E_0 = (\limsup f_n - \liminf f_n)^{-1}(\{0\}) \cap (\cup_{K=1}^{\infty}(\limsup f_n)^{-1}(-K, K)),$$

so E_0 is a measurable set.

(PROBLEM 2)
The conclusion trivially holds if $\varepsilon \geq 1$. Hence we may assume $\varepsilon < 1$.
We first consider the case where $m^*(A) < \infty$. We will prove by contradiction. Assume there exists $\varepsilon_0 \in (0, 1)$ such that $m^*(A \cap I) \leq (1 - \varepsilon_0)m^*(I)$ for any interval I. By the definition of outer measure, one can find a covering of open intervals $\{I_n\}_{n=1}^{\infty}$ such that

$$\cup_{n=1}^{\infty} I_n \supset A \quad \text{and} \quad \sum_{n=1}^{\infty} m^*(I_n) < m^*(A) + \delta, \quad \text{where } \delta = \frac{\varepsilon_0 m^*(A)}{2(1-\varepsilon_0)}.$$

We then have

$$\begin{aligned}
m^*(A) &= m^*(A \cap \cup_{n=1}^{\infty} I_n) \\
&= m^*(\cup_{n=1}^{\infty} A \cap I_n) \\
&\leq \sum_{n=1}^{\infty} m^*(A \cap I_n) && \text{subadditivity} \\
&\leq (1-\varepsilon_0) \sum_{n=1}^{\infty} m^*(I_n) && \text{by our assumption} \\
&< (1-\varepsilon_0) \left(m^*(A) + \frac{\varepsilon_0 m^*(A)}{2(1-\varepsilon_0)} \right) && \text{by our choice of } I_n \\
&= \left(1 - \frac{\varepsilon_0}{2}\right) m^*(A) < m^*(A),
\end{aligned}$$

which is a contradiction.
If $m^*(A) = \infty$, we consider $A_n = A \cap [n, n+1]$, each has finite measure. Since $\cup_{n\in\mathbb{Z}} A_n = A$, we have $m^*(A) \leq \sum_{n\in\mathbb{Z}} m^*(A_n)$. There must exist at least one n such that $m^*(A_n) > 0$. Then by the finite measure case, we can find an interval I such that $m^*(A_n \cap I) > (1-\varepsilon)m^*(I)$, which of course implies $m^*(A \cap I) > (1-\varepsilon)m^*(I)$.

(PROBLEM 3)
We will use the Lebesgue Dominated Convergence Theorem. We let

$$f_n(x) = \frac{nx \ln x}{1 + n^2x^2} \to 0 \text{ as } n \to \infty$$

Also, by the fact that $1 - 2nx + n^2x^2 = (1-nx)^2 \geq 0$, we know

$$\frac{nx}{1+n^2x^2} \leq \frac{1}{2},$$

and therefore $|f_n(x)| \leq \frac{1}{2}|\ln x|$.

$$\int_{[0,1]} |\ln x| dx = -\int_{[0,1]} \ln x dx = [-x \ln x]_0^1 + \int_0^1 x \cdot \frac{1}{x} dx = 1 < \infty.$$

Hence $\ln x$ is Lebesgue integrable over $[0,1]$. By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = \int_{[0,1]} \lim_{n\to\infty} f_n(x)dx = \int_{[0,1]} 0dx = 0.$$

(PROBLEM 4)
We will show for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Since $f_n \to f$ uniformly on $[a,b]$, there exists N such that $\|f_N - f\|_{[a,b]} < \frac{\varepsilon}{4(b-a)}$. As f_N is Riemann integrable on $[a,b]$, there exists a partition P such that $U(f_N,P) - L(f_N,P) < \frac{\varepsilon}{2}$. On each subinterval $I = [x_i, x_{i+1}]$ of the partition P, we have

$$\begin{aligned}
&\sup_I f \leq \sup_I (f - f_N) + \sup_I f_N < \frac{\varepsilon}{4(b-a)} + \sup_I f_N, \text{ and similarly,} \\
&\inf_I f > -\frac{\varepsilon}{4(b-a)} + \inf_I f_N.
\end{aligned}$$

Thus we have $\sup_I f - \inf_I f < \frac{\varepsilon}{2(b-a)} + (\sup_I f_N - \inf_I f_N)$, and therefore,

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{I\in P} (\sup_I f - \inf_I f) \cdot l(I) < \frac{\varepsilon}{2(b-a)} \sum_{I\in P} l(I) + (U(f_N,P) - L(f_N,P)) \\
&< \frac{\varepsilon}{2(b-a)} \cdot (b-a) + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore f is Riemann integrable.

(PROBLEM 5)
Using Egorov's Theorem, for each $k \in \mathbb{N}$, there exists a measurable set $E_k \subset E$ such that $m(E \backslash E_k) < \frac{1}{k}$ and f_n converges uniformly to f on E_k. Let

$$T = E \backslash \left(\cup_{k=1}^{\infty} E_k\right) = \cap_{k=1}^{\infty} E \backslash E_k,$$

then clearly we have

$$E = \left(\cup_{k=1}^{\infty} E_k\right) \cup T.$$

$m(T) = m(\cap_{k=1}^{\infty} E \backslash E_k) \leq m(E \backslash E_k) < \frac{1}{k}$ for each $k \in \mathbb{N}$. Take $k \to \infty$, we have $m(T) = 0$.
The statement is still true for $m(E) = \infty$: let $E_i = E \cap [i, i+1]$ for $n \in \mathbb{Z}$. Each E_i has finite measure and so there exist measurable $E_{i,1}, E_{i,2}, \ldots$ and T_i such that

$$E_i = \left(\cup_{k=1}^{\infty} E_{i,k}\right) \cup T_i,$$

and f_n converges uniformly to f on each $E_{i,k}$ and $m(T_i) = 0$. Since $\cup_{i \in \mathbb{Z}} E_i = E$, we have

$$E = \left(\cup_{i,k=1}^{\infty} E_{i,k}\right) \cup \left(\cup_{i=1}^{\infty} T_i\right).$$

$m(\cup_{i=1}^{\infty} T_i) = 0$ by countable subadditivity.

(PROBLEM 6)
We use the result from PROBLEM 2 (take $\varepsilon = \frac{1}{4}$): there exists an interval I such that $m(A \cap I) > \frac{3}{4} m(I)$. We will show $M \supset (-\frac{1}{3} m(I), \frac{1}{3} m(I))$. For any $c \in (-\frac{1}{3} m(I), \frac{1}{3} m(I))$, we claim $((A \cap I) + c) \cap (A \cap I) \neq \emptyset$. Suppose otherwise, we consider

$$B := ((A \cap I) + c) \cup (A \cap I).$$

Then by additivity $m(B) = m((A \cap I) + c) + m(A \cap I) = 2m(A \cap I) > \frac{3}{2} m(I)$. However, it is easy to see that $B \subset (I + c) \cup I$ and so $m(B) \leq m(I) + |c| < m(I) + \frac{1}{3} m(I) = \frac{4}{3} m(I)$. Hence we have,

$$\frac{3}{2} m(I) < m(B) < \frac{4}{3} m(I),$$

which is clearly a contradiction.
Hence, for any $c \in (-\frac{1}{3} m(I), \frac{1}{3} m(I))$, we have $((A \cap I) + c) \cap (A \cap I) \neq \emptyset$. In other words, there exists $a, b \in (A \cap I)$ such that $a + c = b$, or $c = b - a$. Hence $c \in M$. Therefore M contains the interval $(-\frac{1}{3} m(I), \frac{1}{3} m(I))$.