integration on the real line, we will see that the smallest σ-algebra of sets of real numbers that contains the open sets is a natural object of study.

Definition *The collection $\mathcal{B}$ of Borel sets of real numbers is the smallest σ-algebra of sets of real numbers that contains all of the open sets of real numbers.*

Every open set is a Borel set and since a σ-algebra is closed with respect to the formation of complements, we infer from Proposition 4 that every closed set is a Borel set. Therefore, since each singleton set is closed, every countable set is a Borel set. A countable intersection of open sets is called a G_δ set. A countable union of closed sets is called an F_σ set. Since a σ-algebra is closed with respect to the formation of countable unions and countable intersections, each G_δ set and each F_σ set is a Borel set. Moreover, both the lim inf and lim sup of a countable collection of sets of real numbers, each of which is either open or closed, is a Borel set.

PROBLEMS

27. Is the set of rational numbers open or closed?
28. What are the sets of real numbers that are both open and closed?
29. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.
30. A point x is called an **accumulation point** of a set E provided it is a point of closure of $E \sim \{x\}$.
    (i) Show that the set E' of accumulation points of E is a closed set.
    (ii) Show that $\overline{E} = E \cup E'$.
31. A point x is called an **isolated point** of a set E provided there is an $r > 0$ for which $(x - r, x + r) \cap E = \{x\}$. Show that if a set E consists of isolated points, then it is countable.
32. A point x is called an **interior point** of a set E if there is an $r > 0$ such that the open interval $(x - r, x + r)$ is contained in E. The set of interior points of E is called the **interior** of E denoted by int E. Show that
    (i) E is open if and only if $E = \text{int}\, E$.
    (ii) E is dense if and only if $\text{int}(\mathbf{R} \sim E) = \emptyset$.
33. Show that the Nested Set Theorem is false if F_1 is unbounded.
34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.
35. Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.
36. Show that the collection of Borel sets is the smallest σ-algebra that contains intervals of the form $[a, b)$, where $a < b$.
37. Show that each open set is an F_σ set.

1.5 SEQUENCES OF REAL NUMBERS

A **sequence** of real numbers is a real-valued function whose domain is the set of natural numbers. Rather than denoting a sequence with standard functional notation such as $f : \mathbf{N} \to \mathbf{R}$, it is customary to use subscripts, replace $f(n)$ with a_n, and denote a sequence

49. Let f and g be continuous real-valued functions with a common domain E.
    (i) Show that the sum, $f+g$, and product, fg, are also continuous functions.
    (ii) If h is a continuous function with image contained in E, show that the composition $f \circ h$ is continuous.
    (iii) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.
    (iv) Show that $|f|$ is continuous.
50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.
51. A continuous function φ on $[a, b]$ is called **piecewise linear** provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ for which φ is linear on each interval $[x_i, x_{i+1}]$. Let f be a continuous function on $[a, b]$ and ϵ a positive number. Show that there is a piecewise linear function φ on $[a, b]$ with $|f(x) - \varphi(x)| < \epsilon$ for all $x \in [a, b]$.
52. Show that a nonempty set E of real numbers is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.
53. Show that a set E of real numbers is closed and bounded if and only if every open cover of E has a finite subcover.
54. Show that a nonempty set E of real numbers is an interval if and only if every continuous real-valued function on E has an interval as its image.
55. Show that a monotone function on an open interval is continuous if and only if its image is an interval.
56. Let f be a real-valued function defined on $\mathbf{R}$. Show that the set of points at which f is continuous is a G_δ set.
57. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbf{R}$. Show that the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets.
58. Let f be a continuous real-valued function on $\mathbf{R}$. Show that the inverse image with respect to f of an open set is open, of a closed set is closed, and of a Borel set is Borel.
59. A sequence $\{f_n\}$ of real-valued functions defined on a set E is said to converge uniformly on E to a function f if given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set E. Prove that if $\{f_n\}$ converges uniformly to f on E, then f is continuous on E.
60. Prove Proposition 21. Use this proposition and the Bolzano-Weierstrass Theorem to provide another proof of the Extreme Value Theorem.

PROBLEMS

In the first three problems, let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$

1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.
2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.
3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$.
4. A set function c, defined on all subsets of $\mathbf{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**.

2.2 LEBESGUE OUTER MEASURE

Let I be a nonempty interval of real numbers. We define its length, $\ell(I)$, to be ∞ if I is unbounded and otherwise define its length to be the difference of its endpoints. For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover A, that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of the terms. We define the **outer measure**[3] of A, $m^*(A)$, to be the infimum of all such sums, that is

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

It follows immediately from the definition of outer measure that $m^*(\emptyset) = 0$. Moreover, since any cover of a set B is also a cover of any subset of B, outer measure is **monotone** in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Example A countable set has outer measure zero. Indeed, let C be a countable set enumerated as $C = \{c_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$. For each natural number k, define $I_k = (c_k - \epsilon/2^{k+1},\ c_k + \epsilon/2^{k+1})$. The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C. Therefore

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

This inequality holds for each $\epsilon > 0$. Hence $m^*(E) = 0$.

Proposition 1 *The outer measure of an interval is its length.*

[3]There is a general concept of outer measure, which will be considered in Part III. The set function m^* is a particular example of this general concept, which is properly identified as Lebesgue outer measure on the real line. In Part I, we refer to m^* simply as outer measure.

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{1\leq k,i<\infty} \ell(I_{k,i}) = \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} \ell(I_{k,i})\right]$$

$$< \sum_{k=1}^{\infty}\left[m^*(E_k) + \epsilon/2^k\right]$$

$$= \left[\sum_{k=1}^{\infty} m^*(E_k)\right] + \epsilon.$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. The proof is complete. □

If $\{E_k\}_{k=1}^n$ is any finite collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m^*(E_k).$$

This **finite subadditivity** property follows from countable subadditivity by taking $E_k = \emptyset$ for $k > n$.

PROBLEMS

5. By using properties of outer measure, prove that the interval $[0, 1]$ is not countable.
6. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.
7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

8. Let B be the set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.
9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.
10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

2.3 THE σ-ALGEBRA OF LEBESGUE MEASURABLE SETS

Outer measure has four virtues: (i) it is defined for all sets of real numbers, (ii) the outer measure of an interval is its length, (iii) outer measure is countably subadditive, and (iv) outer measure is translation invariant. But outer measure fails to be countably additive. In fact, it is not even finitely additive (see Theorem 18): there are disjoint sets A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B). \tag{3}$$

13. Show that (i) the translate of an F_σ set is also F_σ, (ii) the translate of a G_δ set is also G_δ, and (iii) the translate of a set of measure zero also has measure zero.
14. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.
15. Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

2.4 OUTER AND INNER APPROXIMATION OF LEBESGUE MEASURABLE SETS

We now present two characterizations of measurability of a set, one based on inner approximation by closed sets and the other on outer approximation by open sets, which provide alternate angles of vision on measurability. These characterizations will be essential tools for our forthcoming study of approximation properties of measurable and integrable functions.

Measurable sets possess the following **excision property**: If A is a measurable set of finite outer measure that is contained in B, then

$$m^*(B \sim A) = m^*(B) - m^*(A). \tag{7}$$

Indeed, by the measurability of A,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^C) = m^*(A) + m^*(B \sim A),$$

and hence, since $m^*(A) < \infty$, we have (7).

Theorem 11 *Let E be any set of real numbers. Then each of the following four assertions is equivalent to the measurability of E.*
(Outer Approximation by Open Sets and G_δ Sets)

(i) For each $\epsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \sim E) < \epsilon$.
(ii) There is a G_δ set G containing E for which $m^*(G \sim E) = 0$.

(Inner Approximation by Closed Sets and F_σ Sets)

(iii) For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \sim F) < \epsilon$.
(iv) There is an F_σ set F contained in E for which $m^*(E \sim F) = 0$.

Proof We establish the equivalence of the measurability of E with each of the two outer approximation properties (i) and (ii). The remainder of the proof follows from De Morgan's Identities together with the observations that a set is measurable if and only if its complement is measurable, is open if and only if its complement is closed, and is F_σ if and only if its complement is G_δ.

Assume E is measurable. Let $\epsilon > 0$. First consider the case that $m^*(E) < \infty$. By the definition of outer measure, there is a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ which covers E and for which

$$\sum_{k=1}^\infty \ell(I_k) < m^*(E) + \epsilon.$$

Define $\mathcal{O} = \bigcup_{k=1}^\infty I_k$. Then $\mathcal{O}$ is an open set containing E. By the definition of the outer measure of $\mathcal{O}$,

$$m^*(\mathcal{O}) \le \sum_{k=1}^\infty \ell(I_k) < m^*(E) + \epsilon,$$