MATH 172 HOMEWORK 1 - SOLUTION TO SELECTED PROBLEMS

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Problem 1 (Chapter 1, Q35). *Show that the collection of Borel sets $\mathcal{B}$ is the smallest σ-algebra that contains the closed sets.*

Any open set is the complement of a closed set. Therefore, $\mathcal{B}$ is a σ-algebra containing all closed sets. To show $\mathcal{B}$ is the smallest, we let Σ be any σ-algebra containing all closed sets. From the definition of σ-algebra, Σ contains all open sets. By the definition of $\mathcal{B}$ in P.20, we have $\mathcal{B} \subset \Sigma$. Hence $\mathcal{B}$ is the smallest among all such Σ.

Problem 2 (Chapter 1, Q37). *Show that each open set is an F_σ set.*

By Proposition 9 of P.17, it suffices to show each open interval is an F_σ. For $a, b \neq \pm\infty$, we have

$$(a,b) = \bigcup_{k=1}^{\infty} \left[a + \frac{1}{k}, b - \frac{1}{k}\right],$$

which is an F_σ set. Note that $(-\infty, b) = (-\infty, b-1] \cup (b-1, b)$ and $(a, \infty) = (a, a+1) \cup [a+1, \infty)$, so they are both F_σ sets. $(-\infty, \infty)$ is itself closed. Hence every open interval is an F_σ set.

Problem 3 (Chapter 1, Q56*). *Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set.*

Let S be the set of points at which f is continuous. We will show instead its complement S^c is an F_σ set. Recall

$$f^*(x_0) := \limsup_{x \to x_0} f(x) = \inf_{\varepsilon > 0} \sup_{|y - x_0| < \varepsilon} f(y)$$
$$f_*(x_0) := \liminf_{x \to x_0} f(x) = \sup_{\varepsilon > 0} \inf_{|y - x_0| < \varepsilon} f(y)$$

Recall f is continous at x_0 if and only if $f^*(x_0) = f_*(x_0)$. Therefore,

$$\begin{aligned} S^c &= \{x : f_*(x) < f^*(x)\} \\ &= \{x : \exists p, q \in \mathbb{Q} \text{ such that } f_*(x) \leq p < q \leq f^*(x)\} \\ &= \bigcup_{p,q \in \mathbb{Q}, p<q} (\{x : f_*(x) \leq p\} \cap \{x : f^*(x) \geq q\}) \end{aligned}$$

Clearly the above union is a countable union. Therefore it suffices to show the sets $\{x : f_*(x) \leq p\}$ and $\{x : f^*(x) \geq q\}$ are closed for each $p, q \in \mathbb{Q}$.
We will show $\{x : f_*(x) \leq p\}$ is closed only, as $f^* = -(-f)_*$.
We need to show $\{x : f_*(x) > p\}$ is open. Given any x_0 such that $f_*(x_0) > p$, by the definition of f_*, there exists $\varepsilon > 0$ such that

$$\inf_{|y - x_0| < \varepsilon} f(y) > p \tag{0.1}$$

We claim $(x_0 - \varepsilon, x_0 + \varepsilon) \subset \{x : f_*(x) > p\}$: for each $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, there exists $\varepsilon' > 0$ (depending on x_0, x and ε) such that $(x - \varepsilon', x + \varepsilon') \subset (x_0 - \varepsilon, x_0 + \varepsilon)$. By (0.1), we have $\inf_{|y - x'| < \varepsilon'} f(y) \geq \inf_{|y - x_0| < \varepsilon} f(y) > p$. Hence $f_*(x) \geq \inf_{|y - x'| < \varepsilon'} f(y) > p$. It proves our claim.
Thus, $\{x : f_*(x) > p\}$ is open, and so $\{x : f_*(x) \leq p\}$ is closed.

Problem 4 (Chapter 2, Q3)**.** *Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\cup_{k=1}^{\infty}) \leq \sum_{k=1}^{\infty} m(E_k)$.*

Let $F_1 = E_1$ and $F_n = E_n \setminus \cup_{k=1}^{n-1} E_k$ for $n \geq 2$. Then clearly $\{F_n\}_{n=1}^{\infty}$ is a countable disjoint collection of sets in $\mathcal{A}$ and $\cup_{k=1}^{\infty} E_k = \cup_{k=1}^{\infty} F_k$. Hence

$$\begin{aligned} m(\cup_{k=1}^{\infty} E_k) &= m(\cup_{k=1}^{\infty} F_k) \\ &= \sum_{k=1}^{\infty} m(F_k) && (F_k\text{'s are disjoint}) \\ &\leq \sum_{k=1}^{\infty} m(E_k) && (F_k \subset E_k \text{ for each } k) \end{aligned}$$

Problem 5 (Chapter 2, Q6)**.** *Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.*

$\mathbb{Q} \cap [0, 1]$ is a countable set. From P.31, we have $m^*(\mathbb{Q} \cap [0, 1]) = 0$. Hence we have

$$1 = m^*([0, 1]) \leq m^*(\mathbb{Q} \cap [0, 1]) + m^*(A) = m^*(A) \leq m^*([0, 1]) = 1.$$

So $m^*(A) = 1$.

Problem 6 (Chapter 2, Q7)**.** *Show that for any bounded set E, there is a G_δ set G for which*

$$E \subset G \text{ and } m^*(G) = m^*(E).$$

By the definition of outer-measure in P.31, for each $n \in \mathbb{N}$, there exists a countable collection of open intervals $\mathcal{I}_n$, such that $E \subset \cup \mathcal{I}_n$ (here $\cup \mathcal{I}_n$ denotes the union of all open intervals in $\mathcal{I}_n$) and

$$m^*(E) \leq l(\mathcal{I}_n) < m^*(E) + \frac{1}{n},$$

where $l(\mathcal{I}_n)$ denotes the sum of lengths of the open intervals in $\mathcal{I}_n$. Note that by countable subadditivity, we have $m^*(\cup \mathcal{I}_n) \leq l(\mathcal{I}_n)$..
Let $G = \cap_{n=1}^{\infty} \cup \mathcal{I}_n$ which is a G_δ set because $\cup \mathcal{I}_n$'s are open. Clearly $E \subset G$. Then we have for each $n \in \mathbb{N}$,

$$m^*(E) \leq m^*(G) \leq m^*(\cup \mathcal{I}_n) \leq l(\mathcal{I}_n) < m^*(E) + \frac{1}{n}.$$

Take $n \to +\infty$, we have $m^*(E) = m^*(G)$.

Problem 7 (Chapter 2, Q10*)**.** *Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.*

Solution 1 (more set-theoretic): We define

$$U = \bigcup_{x \in A} \left(x - \frac{\alpha}{2}, x + \frac{\alpha}{2} \right).$$

Clearly we have $A \subset U$ and, by the condition given in the problem, $B \cap U = \emptyset$. Since U is open, hence measurable, we have

$$m^*(A \cup B) = m^*((A \cup B) \cap U) + m^*((A \cup B) \cap U^c).$$

Since $(A \cup B) \cap U = A$ and $(A \cup B) \cap U^c = B$, we are done.

Solution 2 (more analytic-flavored): Clearly it suffices to show $m^*(A \cup B) \geq m^*(A) + m^*(B)$. We will show for any $\varepsilon > 0$, we have

$$m^*(A \cup B) > m^*(A) + m^*(B) - \varepsilon.$$

Given any $\varepsilon > 0$, there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} l(I_n) < m^*(A \cup B) + \frac{\varepsilon}{2}.$$

For each n, one can 'chop' the open interval I_n so that I_n is a finite union of open subintervals $\{J_{n,i}\}_{i=1}^{K(n)}$ such that $l(J_{n,i}) < \alpha$ and $\sum_{i=1}^{K(n)} l(J_{n,i}) < l(I_n) + \frac{\varepsilon}{2^{n+1}}$. Thus we have,

$$\sum_{n=1}^{\infty} \sum_{i=1}^{K(n)} l(J_{n,i}) < \sum_{i=1}^{\infty} l(I_n) + \frac{\varepsilon}{2}.$$

Let $\mathcal{J}_A = \{J_{n,i} : J_{n,i} \cap A \neq \emptyset\}$ and $\mathcal{J}_B = \{J_{n,i} : J_{n,i} \cap B \neq \emptyset\}$. Then $\mathcal{J}_A$ and $\mathcal{J}_B$ are disjoint collection of open intervals by the fact that $l(J_{n,i}) < \alpha$ and the condition given in the problem. Thus, $\mathcal{J}_A$ and $\mathcal{J}_B$ are, respectively, open covers of A and B. Finally, we have

$$m^*(A \cup B) + \varepsilon > \sum_{n=1}^{\infty} l(I_n) + \frac{\varepsilon}{2} > \sum_{n=1}^{\infty} \sum_{i=1}^{K(n)} l(J_{n,i}) \geq \sum_{J \in \mathcal{J}_A} l(J) + \sum_{J \in \mathcal{J}_B} l(J) \geq m^*(A) + m^*(B).$$

Problem 8 (Chapter 2, Q.14)**.** *Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.*

Let $I_n = [n, n+1]$ for each $n \in \mathbb{Z}$, then $E = \cup_{n \in \mathbb{Z}} E \cap I_n$. We have by countable subadditivity that

$$0 < m^*(E) \leq \sum_{n \in Z} m^*(E \cap I_n).$$

Obviously, at least one of $E \cap I_n$'s has positive outer measure. It is bounded (subset of I_n) and is a subset of E, so it is our desired set.