Problems for the Final

1. Suppose $|z_1| = |z_2| = |z_3| = 1$. Show that z_1, z_2, z_3 are vertices of an equilateral triangle if and only if $z_1 + z_2 + z_3 = 0$.

2. Discuss the curves defined by the following complex functions of a real variable (unless stated otherwise, a, b, ω are real and positive and $-\infty < t < +\infty$): (i) $z = \exp(a + bit)$, (ii) $z = (1 + it)^{-1}$, (iii) $z = at + b\exp(i\omega t)$, (iv) $z = (1 + e^{it})^2$, $0 \le t \le 2\pi$.

3. Suppose that $\mathrm{Re} z_i \ge 0$ and both $\sum z_n$ and $\sum z_n^2$ converge. Show that $\sum |z_n|^2$ converges.

4. Show that any similarity transformation $w = az+b$, $a \ne 0$ may be represented as a composition of a translation, rotation and a dilation with respect to the origin.

5. Prove that a similarity transformation (i) carries circles onto circles, and (ii) parallel straight lines onto parallel straight lines, (iii) leaves the ratio $(z_3 - z_1)/(z_3 - z_2)$ unchanged, (iv) leaves the angle between two curves unchanged.

6. Find all numbers z such that (i) $z^8 = 1$, (ii) $z^2 = 1 + i$, (iii) $\sin z = \sqrt{3}$.

7. (a) Give the definition of equivalence of two paths. (b) Which of the paths (i) $e^{2\pi it}$, $t \in [0, 1]$, (ii) $e^{4\pi it}$, $t \in [0, 1]$, (iii) $e^{-2\pi it}$, $t \in [0, 1]$, and (iv) $e^{4\pi i \sin t}$, $t \in [0.\pi/6]$ are equivalent?

8. (a) Give the definitions of a connected set, and of a path-connected set. (b) Let C be an open connected set. Show that it is path-connected.

9. (a) Give the definitions of an $\mathbb{R}$-linear and $\mathbb{C}$-linear maps. (b) Show that the most general form of an $\mathbb{R}$-linear map is $l(z) = az + b\bar{z}$. (c) Show that the most general form of a $\mathbb{C}$-linear map is $l(z) = az$.

10. (a) Show that a $\mathbb{C}$-linear map preserves angles between vectors. (b) Give an example of an $\mathbb{R}$-linear map that preserves angles between one pair of directions. (c) Show that if an $\mathbb{R}$-linear map preserves orientation and maps some square into a square then it is $\mathbb{C}$-linear.

11. (a) State the Cauchy-Riemann equations. (b) Derive them from the condition $\dfrac{\partial f}{\partial \bar{z}} = 0$.

12. Suppose that f is analytic on an open set A and that $f(z)$ is real for all $z \in A$. Show that $f(z) = \text{const}$.

13. (a) Show that $u(x, y) = x^3 - 3xy^2$ and $v = 3x^2y - y^3$ satisfy the Cauchy-Riemann equations. (b) Find a holomorphic function $f(z)$ such that $f(z) = u(z) + iv(z)$.

14. Suppose that $f(z)$ is a continuous function and that $f(z) = f(2z)$ for all $z \in \mathbb{C}$. Show that $f(z) = \text{const}$.

15. Show that the function $f(z)=|z|^2$ is $\mathbb{C}$-differentiable at $z=0$ but is not analytic anywhere.

16. Show that fractional linear transformations map circles and straight lines into circles or straight lines.

17. Show that if $f(z)=1/z$ then any circle passing through the points $z=-1$ and $z=1$ is mapped onto itself under the map f.

18. Find the linear transformation that maps $z=1$, $z=2$ and $z=3$ into 0, 1 and ∞, respectively.

19. Find a fractional linear transformation which takes the unit disk onto the right half plane with $f(0)=3$.

20. Find fractional linear transformation satisfying $f(z_i)=w_i$, $i=1,2,3$ if $z_1=-1$, $z_2=1$, $z_3=2$ and $w_1=0$, $w_2=-1$ and $w_3=\infty$.

21. Let γ be the circle $|z-a|=R$. Show that

$$\int_\gamma (z-a)^n dz = \begin{cases} 0, & n\neq 1 \\ 2\pi i, & n=1 \end{cases}$$

22. Let $f(z)$ be an $\mathbb{R}$-differentiable function. Show that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int\limits_{|z-a|=\epsilon} f(z)dz = 2\pi i\frac{\partial f}{\partial \bar{z}}(a).$$

23. Let $f(z)$ be holomorphic in an open set D and let $\Delta\subset D$ be a triangle. Show that

$$\int\limits_{\partial\Delta} f(z)dz=0.$$

24. Let $f(z)$ be holomorphic in an open disk $U=\{z:\ |z-a|<R\}$. Show that $f(z)$ has an anti-derivative $F(z)$ inside U.

25. Let D be an open set. a) Define when two paths with the same endpoints are homotopic in D. b) Define when two closed curves are homotopic in D. c) Define when a closed path is homotopic to a point in D. d) Show that if any path in D is homotopic to a point, then any two closed paths are homotopic in D.

26. Let f be analytic in the closure of an open simply connected domain D bounded by a smooth curve. Show that then for all $z\in D$ we have

$$f(z)=\frac{1}{2\pi i}\int\limits_{\partial D}\frac{f(\zeta)}{\zeta-z}d\zeta.$$

27. Use Cauchy's integral formula to show that if f is holomorphic in an open set D then we have for all $z\in D$ and small enough ρ

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi} f\left(z+\rho e^{it}\right)dt.$$

28. Evaluate the following integrals: a) $\int_\gamma \frac{e^z}{z^2}dz$ where γ is the unit circle, (b) $\int_\gamma \frac{z^2-1}{z^2+1}dz$ where γ is a circle of radius 2 centered at 0, (c) $\int_\gamma \frac{1}{z^3}dz$ where γ is the square with vertices $-1-i$, $1-i$, $1+i$, $-1+i$.

29. (a) Suppose F is analytic in a convex domain G and $\mathrm{Re}F'(z) > 0$ for an $z \in G$. Prove that F is univalent in G, that is, $F(z_1) = F(z_2)$ implies $z_1 = z_2$. (b) Show that the function $F(z) = z + e^z$ is univalent in the right half plane.

30. Prove that $\int_0^\pi e^{\cos\theta}\cos(\sin\theta)d\theta = \pi$ by considering $\int_\gamma \frac{e^z}{z}dz$ where γ is the unit circle.

31. Let f be continuous in an open set D. Assume that for every triangle $\Delta \subset D$ we have

$$\int_{\partial\Delta} f(z)dz = 0.$$

Show that $f(z)$ is holomorphic in D.

32. Let γ be the circle $|z| = 2$. Evaluate the following integrals:

$$(a)\ \int_\gamma \frac{dz}{z^2-1},\ (b)\ \int_\gamma \frac{dz}{z^2-8},\ (c)\ \int_\gamma \frac{dz}{z^2+z+1}.$$

33. Let f and g be analytic in a region A, Prove that if $|f| = |g|$ for all $z \in A$ and $f \neq 0$ except at isolated points, then $f(z) = e^{i\theta}g(z)$ for some constant $\theta \in [0, 2\pi]$.

34. Let f be holomorphic in a closed disk $U(z_0; R)$ and $|f(z)| \leq M$ for $z \in \partial U(z_0; R)$. Show that the Taylor coefficients of f at z_0 satisfy $|c_n| \leq M/R^n$.

35. Show that a function that is holomorphic in $\mathbb{C}$ and bounded has to be a constant.

36. Let $p(z)$ be a polynomial. Show that there exists a pointz_0 such that $p(z_0) = 0$.

37. Show that a function f that is holomorphic in $\mathbb{C}$ such that $|f(z)| \geq \pi^\pi$ has to be a constant.

38. Show that derivative of a holomorphic function is holomorphic.

39. Find the radius of convergence of the following power series:

$$(a)\ \sum_{n=0}^\infty nz^n,\ (b)\ \sum_{n=0}^\infty \frac{1}{e^n}z^n,\ (c)\ \sum_{n=0}^\infty \frac{z^n}{1+2^n},\ (d)\ \sum_{n=0}^\infty \frac{z^{2n}}{4^n}.$$

40. Let $f(z)$ be holomorphic in a disk $U(a; R) = \{z : |\ |z-a| < R\}$ and let $f(a) = 0$. Show that there exists a positive integer n such that

$f(z) = (z-a)^n \phi(z)$ where $\phi(z)$ is holomorphic in $U(a;R)$ and $\phi(z) \neq 0$ in a neighborhood of $z = a$.

41. Let f and g be two holomorphic functions in $U(a;R)$ that coincide on a set E that has a limit point in $U(a;R)$. Show that then $f(z) = g(z)$ for all $z \in U(a;R)$.

42. Find the Laurent series expansion of $g(z) = \dfrac{1}{(1+z^2)(2+z^2)}$ for (i) $1 < |z| < \sqrt{2}$, and (ii) $|z| > \sqrt{2}$.

43. Suppose the Laurent series of $f(z) = e^{1/z}/(1-z)$ valid for $0 < |z| < 1$ is $\sum_{n=-\infty}^{\infty} c_n z^n$. Compute c_{-2}, c_{-1}, c_0 and c_1.

44. Let f have a zero at z_0 of multiplicity k. Show that the residue of f'/f at z_0 is k.

45. Find the residues of the following functions at indicated points:

$$(a)\ \frac{e^z - 1}{\sin z},\ z_0 = 0; \qquad (b)\ \frac{e^z + 1}{z^4},\ z_0 = 0;$$
$$(c)\ \left(\frac{\cos z - 1}{z}\right)^2,\ z_0 = 0; \quad (d)\ \frac{z^2}{z^4 - 1},\ z_0 = e^{i\pi/2}.$$

46. Compute the residues of the following functions at their singularities:

$$(a)\ \frac{1}{(1-z)^3}, \quad (b)\ \frac{e^z}{(1-z)^3},$$
$$(c)\ \frac{1}{z(1-z)^3}, \quad (d)\ \frac{e^z}{z(1-z)^3}.$$

47. Evaluate (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 2x + 4}$.

(b) $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{1+x^3} dx$.

(c) $\displaystyle\int_0^{\infty} \frac{\log 2x}{1+x^2} dx$.

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}$.

48. Find the number of roots of the polynomial $p(z) = z^4 + 10z + 1$ in the annulus $1 < |z| < 2$.

49. Let $P(z)$ be a polynomial of degree n and let $M(r) = \sup_{|z|=r} |P(z)|$. Show that $M(r)/r^n$ is a decreasing function.

50. Let $f(z) = u(z) + iv(z)$ be an entire function. Show that its real part $u(z)$ may not attain a local maximum anywhere in $\mathbb{C}$.