Homework #5

1. Replace the condition $f(0) = 0$ in the Schwartz lemma by the condition $f(\alpha) = 0$ ($|\alpha| < 1$) and show that then for $|z| < 1$ the inequality

$$|f(z)| \leq \left| \frac{z - \alpha}{1 - \bar{\alpha} z} \right|$$

is satisfied. Hint: consider the function $g(z) = (1 - \bar{\alpha} z) f(z)/(z - \alpha)$.

2. (i) Let $f(z)$ be holomorphic in the whole complex plane and fix a positive real number $C > 0$. Define $\gamma = \{z \in \mathbb{C} : \ |f(z)| = C\}$, assume that γ is simple closed curve and let D be the domain bounded by the curve γ. Show that there exists a point $z_0 \in D$ such that $f(z_0) = 0$.

(ii) Prove that if $P(z)$ is a polynomial of degree n then for each $C > 0$ the level set of its modulus $K_C = \{\in \mathbb{C} : \ |f(z)| = C\}$ can be decomposed into no more than n connected components.

3. How many roots of the equation

$$z^4 - 8z + 10 = 0$$

are located inside the circle $\{|z| < 1\}$? And in the annulus $\{1 < |z| < 3\}$?

4. Fix $\lambda > 1$ and prove that the equation

$$z = \lambda - e^{-z}$$

has a unique (and hence real) root in the right half-plane.

5. Let none of the zeros of the polynomial

$$P_n(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

lie on the imaginary axis. Prove that when the point z traverses the imaginary axis from below upwards the increase of the argument of $P_n(z)$ equals $k\pi$, where k is an integer of the same parity as n and $|k| \leq n$. Prove also that the polynomials $P_n(z)$ then has $(n + k)/2$ zeros in the right-half plane. Hint: represent $P_n(z)$ as

$$P_n(z) = z^n \left(1 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} \right)$$

and apply the argument principle to the semicircle $|z| < R$, $\text{Re} z > 0$ for a sufficiently large R.

6. Find the residues of the following functions at their singular points:

$$\frac{1}{z^3 - z^5}, \ \sin \frac{z}{z+1}, \ z^3 \cos \frac{1}{z-2}.$$

7. Use residues to evaluate the following integrals:

$$(i) \quad \int_\gamma \frac{z^3 dz}{2z^4+1},$$

where γ is the unit circle $\{|z|=1\}$,

$$(ii) \quad \int_\gamma \frac{z dz}{(z-1)(z-2)^2},$$

where γ is the circle $\{|z-2|=1/2\}$.

8. Find the following definite integrals (it may or may not be useful to set $z=e^{i\phi}$:

$$(i) \int_0^{2\pi} \frac{d\phi}{(a+b\cos\phi)^2},\ a>0, b>0; (ii) \int_0^{2\pi} \frac{d\phi}{1-2a\cos\phi+a^2},\ a\in\mathbb{C}, a\neq\pm 1.$$

9. Use Jordan's lemma to evaluate the following integrals:

$$(i) \int_{-\infty}^{\infty} \frac{x\cos x dx}{x^2-2x+10}, \quad (ii) \quad \int \frac{x\sin x dx}{x^2+4x+20}.$$

10. Compute the integrals

$$\int_0^\infty \frac{x^{1/3}}{1+x^2} dx,$$

and

$$\int_0^\infty \frac{\log x}{1+x^2} dx.$$