Homework #4.

1. Let $f(z)$ be continuous in a disk around z_0. Show that

$$\lim_{r\to 0}\int_{|z-z_0|=r}\frac{f(z)dz}{z-z_0}=2\pi i f(z_0).$$

2. Let $f(z)$ be continuous in $\{|z|\geq 1\}$ and $\lim_{z\to\infty} f(z)=0$. Show that for any $m>0$ and any $a>0$ we have

$$\lim_{R\to\infty}\int_{\Gamma_R} e^{imz}f(z)dz=0.$$

Here $\Gamma_R=\{|z|=R,\mathrm{Im}z\geq a\}$ is an arc of the circle of radius R.

3. Find the integral $\int_\gamma \sin z dz$ from the origin to the point $1+i$ taken along the parabola $y=x^2$.

4. Let $x>0$. Find the limit

$$\lim_{B\to\infty}\int_{-B}^{B}\left(\frac{1}{t+ix}-\frac{1}{t-ix}\right)dt.$$

5. (i) Parametrize the unit circle $\gamma=\{|z|=1\}$ to show that

$$\frac{1}{2\pi i}\int_\gamma\left(z+\frac{1}{z}\right)^n\frac{dz}{z}=\frac{2^n}{2\pi}\int_0^{2\pi}\cos^n t dt.$$

(ii) Expand $(z+\frac{1}{z})^n$ using the binomial formula to show that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2k} t dt=\frac{(2k)!}{2^{2k}(k!)^2}\text{ and }\int_0^{2\pi}\cos^{2k+1} t dt=0.$$

6. Let $p(z)$ be a polynomial and let $\gamma=\{|z-z_0|=R\}$ be a circle. Show, without using the Cauchy integral formula that

$$\frac{1}{2\pi i}\int_\gamma\frac{p(z)}{z-z_0}=p(z_0)$$

and, moreover,

$$\frac{n!}{2\pi i}\int_\gamma\frac{p(z)}{(z-z_0)^{n+1}}=p^{(n)}(z_0).$$

Here $p(^{(n)}(z)$ is the n-th order derivative of $p(z)$.

7. Let γ be a simple closed path and α and β be complex numbers. What are the possible values of

$$\int_\gamma\frac{dz}{(z-\alpha)(z-\beta)}?$$