Homework #2.

1. Prove that if $|z_1| = |z_2| = |z_3|$ then

$$\arg\frac{z_3 - z_2}{z_3 - z_1} = \frac{1}{2}\arg\frac{z_2}{z_1}.$$

2. Verify that $e^x \sin y$ and $e^x \cos y$ are conjugate harmonic functions.

3. Find a conjugate harmonic function v for (a) $u = x^2 - y^2 + xy$, (b) $u = x/(x^2 + y^2)$.

4. Do non-constant harmonic functions of the following form exist: (a) $\phi(xy)$, (b) $\phi(x^2 + y)$, (c) $\phi(x + \sqrt{x^2 + y^2})$? Find a corresponding conjugate function in case it does exist.

5. Find all functions f analytic in $\mathbb{C}\backslash 0$ such that $|f|$ has a constant value on all circles $x^2 + y^2 - ax = 0$.

6. Let $u(x, y)$ and $v(x, y)$ be two conjugate harmonic functions and let $\phi(u, v)$ also be harmonic:

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0.$$

Define $\eta(x, y) = \phi(u(x, y), v(x, y))$ and show that $\eta(x, y)$ is a harmonic function, that is, that

$$\Delta\eta = \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0.$$

Interpret this result in terms of holomorphic functions.

7. (a) Let $f(z)$ be a holomorphic function which maps conformally and one-to-one a domain G onto a domain G'. Find the area of G' in terms of some integral over G.
(b) Find the domain D onto which the function e^z maps the rectangle $1 \le x \le 2$, $0 \le y \le 8$. Calculate the area of this domain using the formula from part (a) and explain why the result is incorrect.

8. Find the image of a circle $|z| = R$ under the mapping $w(z) = z + z^{-1}$.

9. Let $\phi(x, y)$ be a real-valued twice continuously differentiable function with level sets $\Phi_C = \{(x, y) : \ \phi(x, y) = C\}$. Show that for D_C to be the family of level sets of some harmonic function $v(x, y)$ (that is, for each $C \in \mathbb{R}$ there exists $C' \in \mathbb{R}$ so that $D_C = \{(x, y) : \ v(x, y) = C'\}$) it is necessary and sufficient that the ratio $\Delta\phi/|\nabla\phi|^2$ depends only on ϕ. In other words, this ratio is the same for all points $(x, y) \in D_C$ (but the ratio varies when the value of C varies).

10. Show that the most general LFT that maps the upper half space $y \ge 0$ onto itself is given by

$$f(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{R}$.

11. Find the FLT that map the upper half space $y \geq 0$ onto itself, and map $z = 0$ and $z = 1$ onto themselves.

12. Find the FLT that map the circle $|z| = 1$ into the circle $|z - 1| = 1$ such that $z = 0$ and $z = 1$ are mapped to $z = 1/2$ and $z = 0$, respectively.

13. Find a FLT that maps the domain bounded by the circles $|z| = 2$ and $|z - 1| = 1$ into a strip bounded by two straight lines parallel to the imaginary axis.