Some Midterm Solutions

3.(iii) Let G by our open connected set.
Note that $x \sim y$ iff there is a path $\gamma : [0,1] \to G$ with $\gamma(0) = x, \gamma(1) = y$ defines an equivalence relation.

- Symmetry and reflexivity are obvious.
- Transitivity is proven just by concatenating paths.

We claim that each equivalence class is open. Indeed, consider $\{z \in G : z \sim x = [x]$. Since G is open, there is some open ball $B_r(x) \subset G$. But this ball is visibily path connected (it is convex)! Hence, $y \sim x$ for each $y \in B_r(x)$, i.e. $B_r(x) \subset [x]$.
But we know $\sim$ partitions G into disjoint equivalence classes, which are all open. More than one equivalence class would disconnect G. Hence, G contains only one equivalence class, i.e. G is path connected. $\square$

4. We define the supposed inner product via

$$< x, y > := \frac{1}{4}(||x+y||^2 - ||x-y||^2).$$

(i) Positivity $< x, x > = \frac{1}{4}(||x+x||^2 + ||x-x||^2) = ||x||^2 \geq 0$ with equality iff $x = 0$, by positive definitness of the norm $||.||$. So, $< .,. >$ is positive and induces the norm $||.||$.

(ii) Symmetry Since $||z|| = ||-z||$ for all $z \in \mathbb{R}^n$, (and in particular for $z = x - y$) it follows that $< x, y > = < y, x >$ for all $x, y \in \mathbb{R}^n$.

(ii) Additivity

$$\begin{aligned}
4 < x+z, y > &= ||x+z+y||^2 - ||x+z-y||^2 \\
&= 2||z||^2 + 2||x+y||^2 - ||x+y-z||^2 - ||x+z-y||^2 \\
&= 2||z||^2 + 2||x+y||^2 - (2||y-z||^2 + 2||x||^2 - ||y-z-x||^2) - ||x-(y-z)||^2 \\
&= 2||z||^2 + 2||x+y||^2 - 2||y-z||^2 - 2||x||^2 \\
&= 2||z||^2 + \mathbf{2}||\mathbf{y}||^{\mathbf{2}} + 2||x+y||^2 - 2||y-z||^2 - 2||x||^2 - \mathbf{2}||\mathbf{y}||^{\mathbf{2}} \\
&= ||y+z||^2 + ||y-z||^2 - 2||y-z||^2 + 2||x+y||^2 - (||x+y||^2 + ||x-y||^2) \\
&= (||y+z||^2 - ||y-z||^2) + (||x+y||^2 - ||x-y||^2) \\
&= 4 < z, y > + 4 < x, y > .
\end{aligned}$$

(iv) Homogeneity Proving that $< cx, y > = c < x, y >$ for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$ is the last and most difficult part of the proof.
Note that for fixed $x, y \in \mathbb{R}^n$, $f(t) := < tx, y >$ is a continuous function of t which satisfies $f(t_1 + t_2) = f(t_1) + f(t_2)$ (by the additivity which has already been proven). Any such function must satisfy $f(t) = tf(1)$, as you will prove on a subsequent homework assignment (this is where the real work is hiding). Hence, $< cx, y > = f(c) = cf(1) = c < x, y >$, as required.

Thus, $< .,. >$ is indeed an innner product and we are done. $\square$

Remark. The above formula for the inner product is not difficult to think of. Indeed, if you assume the result of the problem before hand, i.e. that the normed vector space $(\mathbb{R}^n, ||.||)$ is actually an inner product space , then $\frac{1}{4}(||x+y||^2 - ||x-y||^2) = \frac{1}{4}(< x,x > +2 < x,y > + < y,y > -(< x,x > -2 < x,y > + < y,y >)) =< x,y >$. Another equivalent way of writing this, using the parallelogram identity, is as $< x,y >:= \frac{1}{2}(||x||^2 + ||y||^2 - ||x-y||^2)$; it is easier to see that this "equals the inner product" but this alternative form is less convenient for calculation and typesetting.

5. (i) Since a_n converges, the sequence $|a_n - \beta|$ is certainly bounded, say $|a_n - \beta| \le R$ for all n. Since a_n converges, choose N_0 such that for all $n \ge N_0$, we have $|a_n - \beta| < \epsilon$.

$$\begin{aligned}
\limsup |s_n - \beta| &\le \limsup \sum_{k=1}^{n} \frac{|a_k - \beta|}{n} \\
&\le \limsup \sum_{k=1}^{N_0} \frac{|a_k - \beta|}{n} + \limsup \frac{\sum_{k=N_0+1}^{n} |a_k - \beta|}{n} \\
&\le \limsup N_0 \times \frac{R}{n} + \limsup \frac{\sum |a_k - \beta|}{n} \\
&\le 0 + \limsup \frac{(n - N_0) \times \epsilon}{n} \\
&\le \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $\limsup |s_n - \beta| = 0$, implying that $\lim_{n\to\infty} s_n = \beta$, as required.

(ii) No, $\lim_{n\to\infty} a_n$ does not necessarily exist if $\lim_{n\to\infty} s_n$ exists. For example, for $a_n = (-1)^n, \lim_{n\to\infty} s_n = 0$ but $\lim_{n\to\infty} a_n$ clearly does not exist.

6. (a) Note that for all $a \in A$ we have $d(x,a) \le d(y,a) + d(x,y)$, by the triangle. Taking the infimum of both sides over all $a \in A$, this shows that $d(x,A) \le d(y,A) + d(x,y)$. Reversing the roles of x and y, this also shows that $d(y,A) \le d(x,A) + d(x,y)$. Combining these two inequalities gives that

$$|f(x) - f(y)| = |d(x,A) - d(y,A)| \le d(x,y).$$

Thus, f is actually uniformly continuous: if $d(x,y) < \epsilon$, then $|f(x) - f(y)| < \epsilon$.
Now note that $A \subset f^{-1}(0) = \{x \in X : f(x) = 0\}$ is closed since f is continuous. Thus, if $f(x) = 0$ implies that $x \in A$, we must have $A = f^{-1}(0)$ is closed.
Conversely, if $x \in A$ and $f(x) = 0$, then there must exist a sequence $a_n \in A$ with $a_n \to x$. But since A is closed, A contains all of its limits points. Thus, $x \in A$.

(b) The function

$$g(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}$$

does the trick. Indeed, by part (a), since A and B are disjoint closed sets, $d(x,A) + d(x,B) > 0$ for all $x \in X$. Thus, g is the ratio of two continuous functions (with the denominator never vanishing), implying that g is continuous. It's also clear that $g|_A = 0, g|_B = 1$. □

7. Let $S_k := \{10^k \le n < 10^{k+1} : n \text{ does not contain any 9's in its base 10 expansion}\}$. Then every $n \in S_k$ can be uniquely expressed in the form $10n' + r$ for some $n' \in S_{k-1}$ and $r \in \{0, 1, ..., 9\}$. Thus,

$$
\begin{aligned}
\sum_{n\in S_k}\frac{1}{n} &= \sum_{n'\in S_{k-1}} \\
&= \sum_{n'\in S_{k-1}}\sum_{r=0}^{8}\frac{1}{10n'+r} \\
&\leq \sum_{n'\in S_{k-1}}\sum_{r=0}^{n}\frac{1}{10n'} \\
&= \frac{9}{10}\sum_{n'\in S_{k-1}}\frac{1}{n'}
\end{aligned}
$$

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Thus, we may iterate this inequality to get $\sum_{n\in S_k}\frac{1}{n}\leq(\frac{9}{10})^k S_0$. Hence,

$$
\begin{aligned}
\sum_{n\in A}\frac{1}{n} &= \sum_{k=0}^{\infty}\sum_{n\in S_k}\frac{1}{n} \\
&\leq \sum_{k=0}^{\infty}(\frac{9}{10})^k S_0 \\
&= 10S_0 \\
&< \infty.
\end{aligned}
$$

This completes the proof. $\square$