Some Assignment 1 Solutions

The T's Problem

To be precise,

Definition of a T. *A "T" is some subset $S \subset \mathbb{R}^2$ such that $S = \overline{AB} \cup \overline{CD}$ where $A, B, C, D \in \mathbb{R}^2$ are distinct points such that*

(i) C lies in the interior of the segment $\overline{AB}$, and

(ii) $AB \perp CD$.

Call the segments $\overline{AC}, \overline{BC}$, and $\overline{DC}$ the stems *of S. Call the points A, B the* endpoints *of S, call D its* bottom (endpoint) *and C its* center.

Suppose we are given a collection $\mathcal{C}$ of T's in the plane such that for each pair $X, Y \in \mathcal{C}$, we have $X \cap Y = \emptyset$. We will prove that C is countable.
It is cumbersome to deal with the arbitrary collection, $\mathcal{C}$, of T's. So, our strategy will be

- To find a convenient countable set of subcollections $\mathcal{C}_i \subset \mathcal{C}, \bigcup_{i \in \mathbb{N}} \mathcal{C}_\rangle = \mathcal{C}$.
- Prove that each $\mathcal{C}_i$ is countable.

Define $\mathcal{C}_n = \{S \in \mathcal{C} : \text{each stem of } S \text{ has length} \geq 1/n\}$. For any $S \in \mathcal{C}$, we can find some $n \in \mathbb{N}$ such that $1/n$ is less than all of its stem lengths. Hence, $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n = \mathcal{C}$.

We proceed to show that each $\mathcal{C}_n$ is countable.
Suppose $S \in \mathcal{C}_n$ with $S = \overline{AB} \cup \overline{CD}$ with A, B, C, D as in the above definition of a T. Since each stem length is $\geq 1/n$, for each $X \in \{A, B, C\}$, there is a unique points X' on the segment $\overline{XC}$ such that $|\overline{X'C}| = 1/n$. Let S' be the tree with endpoints A', B', C'.
Let $\mathcal{D}_n = \{S' : S \in \mathcal{C}_n\}$. Then the map

$$f_n : \mathcal{C}_n \to \mathcal{D}_n : S \mapsto S'$$

is obviously a bijection. Furthermore, since none two T's in $\mathcal{C}_n$ intersect, no two T's in $\mathcal{D}_n$ do either.
Next, consider the map

$$g_n : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto 100n(x, y).$$

Letting $\mathcal{E}_n = \{g_n(S') : S' \in \mathcal{D}_n\}$, we see that

$$h_n : \mathcal{D}_n \to \mathcal{E}_n : S' \mapsto g_n(S')$$

is also a bijection. Again, no two T's in $\mathcal{E}_n$ intersect eachother.
Since $f_n^{-1} \circ g_n^{-1} : \mathcal{E}_n \to \mathcal{C}_n$ is a bijection, it is sufficient to prove that $\mathcal{E}_n$ is countable.

Note that every element of $\mathcal{E}_n$ is a T all of whose stem lengths are 100. The endpoints and center of every such T lie in some unique lattice square, a set of the form $L_{(m,n)} : [n, n+1) \times [m, m+1) : m \in \mathbb{Z}$. Consider the map $h : \mathcal{E}_n \to \mathbb{Z}^2 : S \mapsto (P, Q)$, where the center of S lies in L_P and the bottom of S lies in L_Q. We claim that the non-intersection condition forces this map to be injective.

- Suppose $h(S_1) = (P_1, Q_1) = (P_2, Q_2) = h(S_2)$. So $||P_1 - P_2||, ||Q_1 - Q_2|| <= \sqrt{2}$. We prove that any two T's which satisfy these distance constrains must have non-empty intersection. WLOG, one T has center $(0, 0)$ and bottom $(0, -100)$. Suppose the second T has center $P = (a, b)$ and bottom $Q = (c, d)$.

  Certainly $d < 0$. Hence, if $b \geq 0$, then by the intermediate value theorem, the segment $\overline{PQ}$ must have some point $O = (x, y)$ with $y = 0$. But then x must lie between a and c both of which lie in the interval $(-100, 100)$. Hence, $O \in S_1 \cap S_2$.

- Suppose now that $b \leq 0$. WLOG, we may also assume that $a \geq 0$.
  It is easily computed that the other two endpoints are $A = (x, y) = (d - b + a, b + a - c)$ and $B = (x', y') = (-d + b + a, b - a + c)$, so $x + 100, x' - 100, y, y' \in [-3\sqrt{2}, 3\sqrt{2}]$. Also, since $a \geq 0$ and (a, b) is the midpoint of A and B, we must have $0 \leq x + 100 \leq 3\sqrt{2}$. Thus, it is clear that if $y \geq 0$, then $\overline{AP}$ intersects $[-100, 100] \times \{0\}$, and if $y < 0$, then $\overline{AP}$ intersects $\{0\} \times [-100, 0]$. Hence, $S_1 \cap S_2 \neq \emptyset$ in this case as well.

In all cases, we have $S_1 \cap S_2 \neq \emptyset$. But our assumption tells us that *distinct* T's have empty intersection. The only possible conclusion is, $S_1 = S_2$.
We conclude that $h : \mathcal{E}_n \to \mathbb{Z}^2$ is injective. So, $h \circ g_n \circ f_n : \mathcal{C}_n \to \mathbb{Z}^2$ is injective. Since $\mathbb{Z}^2$ is countable, this implies that $\mathcal{C}_n$ is countable for each n.
We conclude that $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ is countable.

Remark. If we let $S_{m,n}$ denote the T with center at (m, n), bottom at $(m, n - 1/3)$, and all stem lengths equal, then $\mathcal{C} = \{S_{m,n} : (m, n) \in \mathbb{Z}^2\}$ is a countably infinite collection of pairwise disjoint T's in the plain. Thus, this "upper bound" may be attained.

Non-Overlapping Disks

We make use of two basic facts:

(i) If $A \subset B \subset \mathbb{R}^2$ are (measurable) sets, then $Area(A) \leq Area(B)$.

(ii) If $A_1, ..., A_n$ is a finite collection of (measurable) pairwise disjoint sets, then

$$Area(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} Area(A_i).$$

Let $\mathcal{C}$ denote our collection of disks.
Suppose that $D_1, ..., D_k \in \mathcal{C}$ are k closed disks of radius $\geq 1/n$ are contained in another closed disk D of radius M. Then using the familiar formula for the area of a disk and the above two facts, we obtain that

$$k \times \pi \frac{1}{n^2} \leq \sum_{i=1}^{k} Area(D_i) = Area(\bigcup_{i=1}^{n} D_i) \leq Area(D) = \pi M^2.$$

Rearranging gives that $k \leq (nM)^2$.

Let D_M denote the closed disk of radius M centered at the origin. Now, define

$$\mathcal{C}_m = \{D \in \mathcal{C} : D \subset D_m\} \text{ and } \mathcal{C}_{m,n} = \{D \in \mathcal{C}_m : D \subset D_m, radius(D) \geq 1/n\}.$$

We can certainly say that

$$\mathcal{C} = \bigcup_{m\in\mathbb{N}} \mathcal{C}_m = \bigcup_{(m,n)\in\mathbb{N}^2} \mathcal{C}_{m,n}.$$

But by our above calculation, each $C_{n,m}$ is finite (of size $\leq (mn)^2$). Hence, $\mathcal{C}$ is a countable union of finite sets, and so is countable. □

Non-Overlapping Figure 8's

Let $\mathcal{C}$ denote our collection of figure 8's.
Each figure 8, E, is the boundary of two tangent, closed disks. Pick two points, P_E, Q_E, one lying in the interior of each disk, so that both coordinates of P_E and both coordinates of Q_E are rational.

We claim that the map $f : \mathcal{C} \to \mathbb{Q}^2 : E \mapsto (P_E, Q_E)$ is injective.

- Indeed, suppose that $E_1, E_2 \in \mathcal{C}$ are two distinct figure 8's with $(P_{E_1}, Q_{E_1}) = (P_{E_2}, Q_{E_2})$. Suppose E_1 is the boundary of two tangent closed disks, $P_1 \in D$ and $Q_1 \in F$. E_1 and E_2 do not overlap, by assumption. Thus, since the boundary of D disconnects the plane, E_2 must either lie entirely on the interior or entirely on the exterior of D. The same statement holds for F. If it lies exterior to $D \cup F$, then E_1 must be contained in the interior of one of the disks defining E_2. Otherwise, E_2 lies in the interior of either D or F. Whichever case holds, we see that one of the E_i lies in the interior of one of the disks determining the other. This clearly contradicts our assumption

$$\{P_1, Q_1\} = \{P_2, Q_2\}.$$

Hence, f is injective. Since $\mathbb{Q}^2$ is countable, it follows that $\mathcal{C}$ is countable. □

Remark. All three of the posed problems can actually be solved by the same "rational point" method, i.e. associating a finite number of rational points to each non-overlapping figure, which uniquely determines the figure. But as they say, variety is the spice of life.

Other Selected Problems

1.2.1 a.

$$\begin{aligned}
\lambda^2 - \lambda - 2 &= (\lim_{n\to\infty} x_n)^2 - \lim_{n\to\infty} x_n - 2 \\
&= \lim_{n\to\infty} x_n^2 - \lim_{n\to\infty} x_n - 2 \\
&= \lim_{n\to\infty} \sqrt{x_{n-1}+2}^2 - \lim_{n\to\infty} x_n - 2 \\
&= \lim_{n\to\infty} (x_{n-1}+2) - \lim_{n\to\infty} x_n - 2 \\
&= \lim_{n\to\infty} (x_{n-1}+2 - x_n - 2) \\
&= \lim_{n\to\infty} (x_{n-1} - x_n) \\
&= 0.
\end{aligned}$$

b. The two roots of the equation from a. are 2 and -1. But $x_n \geq 0$ for all n. Hence, $|x_n - (-1)| = x_n + 1 \geq 1$, for all n. So, the sequence x_n does not satisfy the definition of convergence to -1. Since we know the limit exists and since the limit is a root of $\lambda^2 - \lambda - 2$, we conclude that

$$2 = \lim_{n\to\infty} x_n.$$

1.4.4. Since $\{x_n\}$ is Cauchy, we can find an N with $1/N < \epsilon/2$ such that $n, m \geq N$ implies that

$$|x_n - x_m| < \epsilon/2.$$

By our assumption, we can find $m > 1/(1/N) = N$ such that $|x_m| < 1/N$. Hence, for all $n \geq N$,

$$|x_n| = |(x_n - x_m) + x_m| \leq |x_n - x_m| + |x_m| < \epsilon/2 + 1/N < \epsilon.$$

Thus, by definition of convergence, $x_n \to 0$.

Chapter 1, 28. Suppose $\lim_{n\to\infty} x_n = L$.
There exists an N such that $n \geq N$ implies that $|x_n - L| < \epsilon/2$. Hence, for all $n \geq N$, we have that

$$L - \epsilon/2 \leq x_n \leq L + \epsilon/2.$$

Hence, for all $n \geq N, L + \epsilon/2$ is an upper bound and $L - \epsilon/2$ is a lower bound for the set $\{x_k : k \geq n\}$. Hence,

$$L - \epsilon/2 \leq A_n \leq B_n \leq L + \epsilon/2.$$

Hence, A_n, B_n, and L all lie in the interval $[L - \epsilon/2, L + \epsilon/2]$ for all $n \geq N$. This implies that for all $n \geq N, |A_n - L|, |B_n - L| < \epsilon$. Hence,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = L,$$

as required.