Homework # 5. Extra problems.

1. Show that a function monotonic on $[0,1]$ has at most countably many discontinuities.

2. The ternary expansion of a real number $r \in [0,1]$ is its representation as $r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n = 0, 1, 2$. The sequence $\{a_n\}$ is called the ternary expansion of r. Such an expansion is unique (up to a tail of two's) for a given $r \in [0,1]$. Recall from Exercise 39 on p. 176 in Marsden, Hoffman, that the ternary Cantor set C consists of all real numbers in $[0,1]$ such that $a_n \neq 1$ for all $n \geq 1$ (if r has two ternary expansions, we put it into C if one of the expansions contains no ones).

(i) Show that the Cantor set is uncountable.

(ii) Let $x \in [0,1]$ have a ternary expansion $\{a_n\}$. Let $N = \infty$ if none of a_n are 1, otherwise let N be the smallest value of n such that $a_n = 1$. Let $b_n = a_n/2$ for $n < N$ and $b_N = 1$. Show that the sum $f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}$ is independent of the ternary expansion if x has two expansions.

(iii) Show that $f(x)$ is a monotone continuous function on $[0,1]$ that is constant on each interval contained in the complement of the Cantor set.

(iv) Let $f(x)$ be the Cantor function, and let $g(x) = f(x) + x$. Show that g is a homeomorphism (g^{-1} is continuous) of $[0,1]$ onto $[0,2]$.

(v) Show that the Cantor function is continuous but not absolutely continuous.

3. A function f is *lower semi-continuous* at a point y if $f(y) \leq \liminf_{x \to y} f(x)$. Show that a function f is lower semi-continuous if and only if the set $\{f(x) > \lambda\}$ is open for all $\lambda \in \mathbb{R}$,