Assignment 7: Some Extra Problem Solutions

1. – We prove that S^c is open.
   Suppose $x \in S^c$ so that $\sum_{k=1}^{\infty} k^2 x_k^2 > 1$. Then there is some N such that $\sum_{k=1}^{N} k^2 x_k^2 > 1$. But the function $f : \mathbb{R}^n \to \mathbb{R} : (x_1, ..., x_n) \mapsto \sum_{k=1}^{N} k^2 x_k^2$ is a polymonial function and so is certainly continuous. Hence, $f^{-1}((1,\infty))$ is open, i.e. for every $x \in f^{-1}((1,\infty))$, there is some $B_\delta(x) \subset f^{-1}((1,\infty))$. But this means that as long as $\sum_{k=1}^{N} |y_k - x_k|^2 < \delta^2$, then $f(y) > 1$. So certainly, if $\sum_{k=1}^{\infty} |y_k - x_k|^2 < \delta^2$, then $y \in S^c$, or equivalently $B_\delta(x) \subset S^c$. Hence, S^c is open. This implies that $(S, ||.||_2)$ is a complete metric space, being a closed subset of the complete metric space l^2.

   – If $x \in S$, then

$$N^2 \sum_{k>N} x_k^2 \leq \sum_{k>N} k^2 x_k^2 \leq \sum_{k\geq 1} k^2 x_k^2 \leq 1.$$

   Hence, $\sum_{n\geq N} x_k^2 \leq \frac{1}{N^2}$. Also, note that $S'\{x \in \mathbb{R}^N : \sum_{k=1}^{N} k^2 x_k^2\}$ is compact and hence totally bounded. Thus, we can find finitely many points $y_1, ..., y_n \in \mathbb{R}^n$ such that for every $x = (x_1, ..., x_N) \in S', \sum_{k=1}^{N} |y_{ik} - x_k|^2 < \frac{\epsilon^2}{2}$ for some $i = 1, ..., N$. But then the n points $z_i = (y_i, 0, 0, ...) \in S$, form a $\sqrt{\frac{1}{N^2} + \frac{\epsilon^2}{2}}$ net. Thus, as long as $\frac{1}{N} < \frac{\epsilon}{\sqrt{2}}$, we have that $\{z_1, ..., z_n\}$ forms an ϵ net for S. Since ϵ was arbitrary, S is totally bounded.

Thus, S is a complete and totally bounded metric space and so is compact. $\square$