Assignment 6: Extra Problem Solutions

1. We will prove in stages that $f(q) = qf(1)$ for all $q \in \mathbb{Q}$.

   - $f(0) = f(0+0) = f(0) + f(0)$, implying that $f(0) = 0$.
   - $f(x) + f(-x) = f(x + (-x)) = f(0) = 0$, implying that $f(-x) = -f(x)$.
   - If m is a positive integer, $x \in \mathbb{R}$, then $f(mx) = f(x + ..._m + x) = f(x) + ..._m + f(x) = mf(x)$, where $a + ..._m a$ denotes taking the sum m copies of a. Thus, combined with the above, we can say that $f(mx) = mf(x)$ for any $m \in \mathbb{Z}$.
   - If n is a positive integer, then $nf(\frac{x}{n}) = f(\frac{x}{n}) + ..._n + f(\frac{x}{n}) = f(\frac{x}{n} + ..._n + \frac{x}{n}) = f(x)$, i.e. $f(\frac{x}{n}) = \frac{1}{n}f(x)$.
   - Combining the above items, we see that for any $m \in \mathbb{Z}, 1 \leq n \in \mathbb{Z}, x \in \mathbb{R}$, that $f(\frac{m}{n} \times x) = \frac{m}{n}f(x)$. Thus, setting $x = 1$, we get

     $$f(q) = qf(1)$$

     for all $q \in \mathbb{Q}$.

   Since $f(x) - xf(1)$ is continuous, $S = \{x : f(x) - xf(1) = 0\}$ is closed. But the above work shows that $S \supset \mathbb{Q}$. Hence $S = \mathbb{R}$ and $f(x) = xf(1)$ for all $x \in \mathbb{R}$. $\square$

2. Consider the function f defined on $[0, 1]$ by

   $$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some positive integer } n, \\ 0 & \text{otherwise.} \end{cases}$$

   Clearly, f is discontinuous precisely at the points $S = \{\frac{1}{n}\} \cup \{0\}$, a countably infinite set. We claim that f is Riemann integrable.

   - Only finitely many points in S do not lie in $I_0 = [0, \frac{\epsilon}{2}]$, say M of them. Find pairwise disjoint intervals $I_1, ..., I_M$, which are also disjoint from I_0, such that $I_1, ..., I_M$ cover these M points and $|I_1| + ... + |I_M| < \frac{\epsilon}{2}$. The endpoints of the intervals and the point 1 determine a partition $\mathcal{P}$ of $[0, 1]$. By construction, for any refinement $\mathcal{P}'$ of this partition and any Riemann sum $0 \leq S(f, \mathcal{P}') < \epsilon$. Hence, f is Riemann integrable with $\int f = 0$. $\square$

3. Consider $f : [0, 1] \to \mathbb{R}$,

   $$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \cap \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

   Every interval contains both a rational and irrational number. Thus, the lower Riemann sum is 0 for any partition and the upper Riemann sum is 1 for any partition of $[0, 1]$. Hence, f is not Riemann integrable.

   On the other hand, we claim that f is the pointwise limit of Riemann integrable functions.

  - Indeed, we can enumerate $\mathbb{Q} \cap [0, 1]$ as $\{r_n : n \geq 1\}$. Let $S_n = \{r_1, ..., r_n\}$. We have $I_{S_n}(x)$ is Riemann integrable (where I denotes the indicator function). Also $I_{S_n}(x) \to f(x)$ for all $x \in [0, 1]$.

This therefore gives a counterexample. $\square$