

Assignment 5: Extra Problem Solutions

1. Let $f : [0, 1] \rightarrow \mathbb{R}$ be our monotone function. Clearly,

$$\lim_{x \rightarrow x_0^+} f(x) = \inf\{f(x) : x > x_0\} =: f(x_0)^+, \quad \lim_{x \rightarrow x_0^-} f(x) = \sup\{f(x) : x < x_0\} =: f(x_0)^-.$$

In particular, the right and left limits exist. Let S denote the set of discontinuities of f . For each $x_0 \in S$, we assign the (non-empty) open interval $I_{x_0} = (f(x_0)^+, f(x_0)^-)$. Note that if $x_0 < x_1$ are two points of discontinuity, then for any $x_0 < c < x_1$, we have $f(x_0)^+ \leq f(c) \leq f(x_1)^-$, thus implying that I_{x_0} and I_{x_1} are disjoint, with I_{x_0} lying entirely to the left of I_{x_1} . But \mathbb{R} can contain at most a countable family of pairwise disjoint non-empty open intervals (each one contains a distinct element of \mathbb{Q}). Hence, $\{I_{x_0}\}_{x_0 \in S}$ is countable, i.e. S is countable. \square

3. Let X be any metric space and let $f : X \rightarrow \mathbb{R}$ be lower semicontinuous.

Suppose $x : f(x) > \lambda$ for each lambda. Then for any fixed $x_0, \lambda = f(x_0) - \epsilon$, we see that $f(x_0) > \lambda$. Thus, by assumption, there is a neighbourhood $x_0 \in B_\delta(x_0)$ such that $f(z) > f(x_0) - \epsilon$ for all $y \in B_\delta(x_0)$. Then $\inf_{y \in B_\delta(x_0)} f(y) \geq f(x_0) - \epsilon$. Thus, certainly,

$$\limsup_{y \rightarrow x_0} f(y) = \sup_{\delta > 0} \inf_{y \in B_\delta(x_0)} f(y) \geq \inf_{y \in B_\delta(x_0)} f(y) \geq f(x_0) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get that $\limsup_{y \rightarrow x_0} f(y) \geq f(x_0)$.

Conversely, suppose $f(x_0) > \lambda$, say $f(x_0) - \lambda > \epsilon > 0$. By assumption, $\sup_{\delta > 0} \inf_{y \in B_\delta(x_0)} f(y) \geq f(x_0)$. Hence, by definition of the supremum, there is some $\delta > 0$ for which $\inf_{y \in B_\delta(x_0)} f(y) \geq f(x_0) - \epsilon > \lambda$. Thus, $x_0 \in B_\delta(x_0) \subset \{x_0 : f(x) > \lambda\}$. It follows that $\{x : f(x) > \lambda\}$ is open. \square