Assignment 3: Some Extra Problem Solutions

1. (i) The sequence $1/n \to 0$ monotonically. Thus, it follows that S_n converges by the Alternating Series Test.

   (ii) Fix any $c \in \mathbb{R}$. Our strategy is to "oscillate around c with small differences". We outline an algorithm for how to arrange the series. We do this iteratively, constructing the set of positive terms P_n, and the set of negative terms N_n involved in the n^{th} partial sum.

   - ∗ Start with sets $P_0, N_0 = \emptyset, R_0^+ = R_0^- = \{1, 1/3, 1/4, ...\} \cup \{-1/2, -1/4, ...\}$.
   - ∗ If $\sum_{a \in P_n \cup N_n} a \leq c$, then let P_n' = the k positive terms of greatest absolute value in R_n, where k is the least positive integer such that $\sum_{a \in P_n'} + \sum_{a \in P_n \cup N_n} a > c$. Such a positive integer k must exist since the sum of positive terms diverges (by Comparison with the harmonic series). Then let $P_{n+1} = P_n \cup P_n', N_{n+1} = N_n, R_{n+1}^+ = R_n - P_n'$.
   - ∗ If $\sum_{a \in P_n \cup N_n} a \geq c$, then let N_n' = the k negative terms of greatest absolute value in R_n, where k is the least positive integer such that $\sum_{a \in N_n'} + \sum_{a \in P_n \cup N_n} a < c$. Such a positive integer k must exist since the sum of negative terms diverges. Then let $P_{n+1} = P_n, N_{n+1} = N_n \cup N_n', R_{n+1}^- = R_n - N_n'$.
   - ∗ Let $R_n = R_{n+1}^+ \cap R_{n+1}^-$.

   Arrange the terms in blocks $P_0'N_0'...P_n'N_n'...$ or $P_0'N_0'...P_n'N_n'...$ (in the respective cases $c \geq 0$ or $c < 0$) where the terms within the P_n' and N_n' blocks are arranged in decreasing order of absolute value.

   - ∗ Each P_n' and N_n' block is non-empty and they use up the positive (resp. negative) terms in decreasing order of absolute value. Hence, this blocking determines a permutation of the original series.
   - ∗ By the minimality of the integer k in our definition of the P_n' (resp. N_n',) we have that

     $$\left| \left(\sum_{a \in S} a + \sum_{a \in P_n \cup N_n} a \right) - c \right| \leq \min_{a \in R_n} |a|$$

     for any subset $S' \subset P_n'$ (resp. N_n'). Thus, as long as $m \geq \sum_{k=0}^{n} |P_k'| + |N_k'|$, we get that

     $$|S_m - c| \leq \min_{a \in R_n} |a|.$$

     But certainly, since at least n positive terms and n negative terms are exhausted in $P_n \cup N_n$. Hence,

     $$\min_{a \in R_n} |a| \leq 1/n \to 0.$$

   It follows that $\lim_{n \to \infty} S_n = c$, as required. □

   Remark. The key to the above proof lies in the fact that the series $\sum_{j=1}^{\infty} (-1)^{j+1}/j$ is only conditionally convergent. The statement of the problem is blatantly false for absolutely convergent series, while it is true for arbitrary conditionally convergent series.

2. Define

$$f(x) = \begin{cases} 1 - n^2|x - n^2| & \text{if } |x - n^2| \leq 1/n^2 \text{ for } n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

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$$\int_1^\infty f(x)dx = \sum_{n=2}^\infty 1/n^2 < \infty.$$

– On the other hand, $f(n) = 1$ for all $n \geq 2$. Thus, $\sum_{n=1}^\infty f(n)$ certainly diverges. □