Assignment 2: Some Extra Problem Solutions

1. It is easily verified that for any element $r \in K$, whether or not $r \geq 0$ does not depend on the representation of r as a ratio of polynomials. Let $P = \{r \in K : r > 0\}$ be the positive cone of K. Let $p_1, q_1, p_2, q_2 \in \mathbb{R}[x]$ be any non-zero polynomials satisfying $p_1/q_1, p_2/q_2 \in P$. By our earlier remark, we may assume that the lead coefficient of all of these polynomials are positive.

   - $p_1/q_1 + p_2/q_2 = (p_1q_2 + p_2q_1)/q_1q_2$. The product of two polynomials with a positive lead coefficient is another polynomial with positive lead coefficient. Thus, the denominator has a positive lead coefficient. Since the numerator is a sum of two such polynomials, it also has a positive lead coefficient. Thus, by definition of positivity in $K, p_1/q_1 + p_2/q_2 \in P$.
   - $p_1/q_1 \times p_2/q_2 = p_1p_2/q_1q_2$. By our remarks above, both the numerator and demoninator have positive lead coefficient. Hence, $p_1/q_1 \times p_2/q_2 \in P$.
   - $K = P \cup \{0\} \cup -P$ and the union is disjoint.

   Thus, K is an ordered field.
   However, $1 + ... + 1 = n < x$ for any n. Hence, K is not archimedean.

2.,3. We will construct a function whose range is all of $\mathbb{R}$ on any open subinterval of $[0, 1]$.
   Consider the set of intervals $\mathcal{C} = \{(p, q) : p, q \in \mathbb{Q}, 0 < p, q < 1\}$. Note that every non-empty open interval $(a, b) \subset [0, 1]$ contains an element of $\mathcal{C}$. Thus, it is sufficient to construct a function $f : [0, 1] \to \mathbb{R}$ such that $f((p, q)) = \mathbb{R}$ on each $(p, q) \in \mathcal{C}$. Since $\mathcal{C}$ is countable, we may enumerate it as $I_1, I_2, ..., I_n, ...$
   We will proceed to construct sets $K_j \subset I_j$ such that

   - For each j, there is a bijection $f_j : K_j \to \mathbb{R}$.
   - The K_j are pairwise disjoint.

   This seems a lot to ask. But fortunately, we can find sets K_j sufficiently pathological to do the job.

   For this purpose, we will introduce the Cantor set C. It is defined by an iterative procedure.

   - Let $C_0 = [0, 1]$
   - Suppose $C_n = \bigcup_{\epsilon_i \in \{0,1\}} I_{\epsilon_1 \ldots \epsilon_n}$, where the $I_{\epsilon_1 \ldots \epsilon_n}$ are pairwise disjoint closed intervals. Remove the open middle third from each interval in the union, i.e. if $[a, b]$ is in the union, then we get $[a, b] - (2/3a + 1/3b, 1/3a + 2/3b) = [a, 2/3a + 1/3b] \cup [1/3a + 2/3b, b]$. What remains of each interval is a left closed interval and a right closed interval. Let $I_{\epsilon_1 \ldots \epsilon_n} = I_{\epsilon_1 \ldots \epsilon_n 0} \cup I_{\epsilon_1 \ldots \epsilon_n 1}$ where $I_{\epsilon_1 \ldots \epsilon_n 0}$ and $I_{\epsilon_1 \ldots \epsilon_n 1}$ are respectively the left and right closed subintervals which remain when the middle third of $I_{\epsilon_1 \ldots \epsilon_n}$ is removed. Then, we define

   $$C_{n+1} = \bigcup_{\epsilon_i \in \{0,1\}} I_{\epsilon_1 \ldots \epsilon_{n+1}}.$$

   - Define $C = \bigcap_{n=0}^{\infty} C_n$.

   We proceed to illustrate some properties of C.

   - C is in bijection with all infinite binary sequences.

- * Let $\epsilon_1\epsilon_2...\epsilon_n...$ be one such sequence. Note that $I_{\epsilon_1} \supset I_{\epsilon_1\epsilon_2} \supset I_{\epsilon_1\epsilon_2...\epsilon_n} \supset ...$ is a decreasing sequence of closed intervals with diameter approaching 0. Hence, there is a unique point $x = f(\epsilon_1\epsilon_2...\epsilon_n...) \in \bigcap_{n=1}^{\infty} I_{\epsilon_1\epsilon_2...\epsilon_n}$.
  * Let $x \in C$. Note that any two intervals $I_{\epsilon_1...\epsilon_n} \subset C_n, I_{\delta_1...\delta_m} \subset C_m, n \leq m$ are either disjoint, or $I_{\epsilon_1...\epsilon_n} \subset I_{\delta_1...\delta_m}$. The latter case occurs iff $\epsilon_1...\epsilon_n$ is a prefix of $\delta_1...\delta_m$, i.e. $\epsilon_i = \delta_i$ for each $1 \leq i \leq n$. Since x lies in exactly one constituent interval of C_n for each n, our remarks show that there is a unique sequence $g(x) = \epsilon_1...\epsilon_n...$ such that $x \in \bigcap_{n=1}^{\infty} I_{\epsilon_1...\epsilon_n}$.
  * It is easily seen that our constructions are inverse to eachother, i.e. $f : 2^{\mathbb{N}} \to C$ and $g : C \to 2^{\mathbb{N}}$ are inverse maps, proving our claim.

- C is closed.
  * This is obvious, because C is the intersection of closed sets.

- C has no interior, i.e. does not contain any open interval.
  * If C contains the open interval (a, b), then each C_n does too. But consecutive intervals in C_n contain gaps of size $\geq 3^{-n}$. Choose n such that $b - a > 3^{-n}$. then (a, b) must intersect one of these gaps, i.e. must intersect the complement of C_n. This is a contradiction. Hence C has no interior.

For any closed interval $[a, b] \subset [0, 1]$, we let $C_{[a,b]} := f(C)$ where f is the homeomorphism $f(x) = (b - a)x + a$. $C_{[a,b]}$ has exactly the same properties as C.
Now, we are ready to construct the subsets $K_n \subset I_n = (p_n, q_n)$ alluded to earlier. We do so recursively.

- Let $[a_1, b_1]$ be any closed subinterval of I_1. Let $K_1 = C_{[a_1,b_1]}$. K_1 is closed and has no interior, and there is a bijection $f_1 : K_1 \to \mathbb{R}$.
- Suppose we have constructed $K_j \subset I_j, 1 \leq j \leq n$ with the K_j closed, $\bigcup_{j=1}^{n} K_j$ has no interior, and there is a bijection $f_j : K_j \to \mathbb{R}$ for each $1 \leq j \leq n$.
  I_{n+1} is an open interval. Since $\bigcup_{j=1}^{n} K_j$ is a closed set with no interior, $I_{n+1} - bigcup_{j=1}^{n} K_j$ is a non-empty open set. Hence, we can find some open subinterval (c_{n+1}, d_{n+1}) and closed subinterval $[a_{n+1}, b_{n+1}]$ with

  $$[a_{n+1}, b_{n+1}] \subset (c_{n+1}, d_{n+1}) \subset I_{n+1} - bigcup_{j=1}^{n} K_j.$$

  Let $K_{n+1} = C_{[a_n,b_n]}$. This is disjoint from each $K_j, j \leq n$ and there is a bijection $f_{n+1} : K_{n+1} \to \mathbb{R}$.
  Lastly, in general, if A and B are closed subsets of $\mathbb{R}$ with no interior, then $A \cup B$ has no interior. Indeed, suppose $I \subset \mathbb{R}$ is any non-empty interval. Then since A has no interior, $I - A$ is a non-empty open set. Thus, it contains a non-empty open interval, say I'. Then since B has no interior, $I' - B$ is a non-empty open set. It follows that $I - A \cup B$ is a non-empty open set, and in particular is non-empty, thus proving our claim.
  Thus, by our inductive assumption, $\bigcup_{j=1}^{n+1} K_j$ has no interior.

Ultimately, this creates a sequence K_n which has the desired properties.

Thus, to make our pathological function $h : [0, 1] \to \mathbb{R}$, we define

$$h(x) = \begin{cases} f_n(x) & \text{if } x \in K_n, \\ 0 & \text{otherwise.} \end{cases}$$

This function h already solves question 2.
We modify it slightly to get question 3. Namely, choose any function $k : \mathbb{R} \to [0, 1]$ which is surjective, e.g. $g(y) = 1/2(1 + \sin(y))$. Then $g \circ h$ provides an example for question 3. $\square$