Assignment 7: Some Textbook Problem Solutions

26. Consider the map $T : C[0,1] \to C[0,1] : Tf(x) = A(x) + \int_0^1 k(x,y)f(y)dy$. Since k is continuous on the compact set $U, |k(x,y)|$ assumes its maximum value on U. By assumption, we can thus say that $|k(x,y)| \leq r < 1$ for some $0 \leq r < 1$. Note that

$$\begin{aligned} |Tf(x) - Tg(x)| &= \left| \int_0^1 k(x,y)(f(y) - g(y))dy \right| \\ &\leq \int_0^1 |k(x,y)||f(y) - g(y)|dy \\ &\leq \int_0^1 r||f-g||_\infty dy \\ &= r||f-g||_\infty . \end{aligned}$$

Hence, $||Tf - Tg||_\infty \leq r||f-g||_\infty$. Thus, T is a contraction mapping on the complete metric space $(C[0,1], ||.||_\infty)$. By the Banach contraction mapping principle (proven on an earlier assignment), we have that T has a unique fixed point f. This f is the required function. $\square$

40b. For a fixed $f_0 \in C[0,1]$, we know that $|f(x)| \leq M$ for some $M > 0$ and all $x \in [0,1]$. But we know that g is uniformly continuous on the compact interval $[-M - \epsilon_0, M + \epsilon_0]$. Hence, there exists $\delta > 0$ such that if $|x - y| < \delta$ for $x, y \in [-M - \epsilon_0, M + \epsilon_0]$, then $|g(x) - g(y)| < \epsilon$.
Suppose that $||f - f_0||_\infty < \delta, \epsilon_0$. Then for any fixed $x \in [0,1]$, we know that $|f(x) - f_0(x)| < \delta$ and $|f(x)|, |f_0(x)| \leq M + \epsilon_0$. Hence, by our choice of δ, we know that $|g(f(x)) - g(f_0(x))| < \epsilon$. Since $x \in [0,1]$ was arbitrary, $||g \circ f - g \circ f_0||_\infty < \epsilon$. It follows that F is continuous.

Suppose that g is uniformly continuous. Then there exists δ such that $|x - y| < \delta$ implies that $|g(x) - g(y)| < \epsilon$. Then as long as $||f_1 - f_2||_\infty < \delta$, for any fixed $x \in [0,1], |f_1(x) - f_2(x)| < \delta$. Hence, by choice of delta $|g(f_1(x)) - g(f_2(x))| < \epsilon$. Since $x \in [0,1]$ is arbirary, it follows that $||g \circ f_1 - g \circ f_2||_\infty \leq \epsilon$. Thus, F is uniformly continuous in this case.

49. The proof that $(\mathcal{B}, ||.||_\infty)$, where $||.||_\infty$ is given by $||f||_\infty = \sup_{x \in \mathbb{R}} |f(x)|$ makes $\mathcal{B}$ a normed vector space is completely routine. Thus, we must only show that $(\mathcal{B}, d)$ with distance function defined by $d(f,g) = ||f - g||_\infty$ is a complete metric space.

Suppose that $\{f_n\}$ is a Cauchy sequence in $\mathcal{B}$. Then there exists an N such that if $m, n \geq N$ then $||f_n - f_m||_\infty < \epsilon$. Thus, for any fixed $x \in \mathbb{R}$, we know that $|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty < \epsilon$. It follows that $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$, for each $x \in \mathbb{R}$. Hence, by the completeness of $\mathbb{R}$, we know that $f_n(x) \to f(x)$ for some $f(x) \in \mathbb{R}$. f is at least a candidate limit for f_n.

- Since f_n is a Cauchy sequence, it is bounded with $||f_n||_\infty \leq M$ for some $M > 0$. For fixed $x \in \mathbb{R}, \lim_{n \to \infty} f_n(x) = f(x)$, by construction. Hence, there is some N such that $|f(x) - f_N(x)| \leq 1$, implying that $|f(x)| \leq |f_N(x)| + 1 \leq ||f_n||_\infty + 1 \leq M + 1$. Hence, $||f||_\infty = \sup_{x \in \mathbb{R}} |f(x)| \leq M + 1 < \infty$, i.e. $f \in \mathcal{B}$.

- Choose N such that $n, m \geq N$ implies that $||f_n - f_m||_\infty < \epsilon$. Then for any $n \geq N$ and any fixed $x \in \mathbb{R}$,

  $$\begin{aligned} |f(x) - f_n(x)| &= \lim_{m\to\infty} |f_m(x) - f_n(x)| \\ &\leq \limsup_{m\to\infty} |f_m(x) - f_n(x)| \\ &\leq \limsup_{m\to\infty} ||f_m - f_n||_\infty \\ &\leq \epsilon. \end{aligned}$$

  Since this holds for arbitrary $x \in \mathbb{R}$, we must have $||f - f_n||_\infty = \sup_{x\in\mathbb{R}} |f(x) - f_n(x)| \leq \epsilon$.

It follows that $f_n \to f \in \mathcal{B}$. This proves that $\mathcal{B}$ is a Banach space. □

68. a. We define a metric d on $\mathcal{C}(A, \mathbb{R}^m) \times A$ by $d((f_1, x_1), (f_2, x_2)) = ||f_1 - f_2||_\infty + ||x_1 - x_2||_2$.

    Fix $(f_0, x_0) \in \mathcal{C}(A, \mathbb{R}^m) \times A$. Since f_0 is uniformly continuous on A, there is some $\delta > 0$ such that $||x - y||_2 < \delta$ implies that $||f_0(x) - f_0(y)||_2 < \frac{\epsilon}{2}$. Then as long as $d((f, x), (f_0, x_0)) < \delta, \frac{\epsilon}{2}$, we get that

    $$\begin{aligned} ||f(x) - f_0(x_0)||_2 &\leq ||f(x) - f_0(x)||_2 + ||f_0(x) - f_0(x_0)||_2 \\ &\leq ||f - f_0||_\infty + \frac{epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

    Hence, E is continuous. □

    b. If $\mathcal{B} \subset \mathcal{C}(A, \mathbb{R}^m)$ is compact, then $\mathcal{B} \times A$ is a compact subset of $\mathcal{C}(A, \mathbb{R}^m) \times A$. Hence, the restriction of E to $\mathcal{B} \times A$ is uniformly continuous. This implies that that for any $\epsilon > 0$ there is some $\delta > 0$ such that for every $(f, x_1), (f, x_2) \in \mathcal{B} \times A$ satisfying $d((f, x_1), (f, x_2)) < \delta$, we have that $||E(f(x_1)) - E(f(x_2))|| < \epsilon$. This says precisely that $||x_1 - x_2|| < \delta$ implies that $||f(x_1) - f(x_2)|| < \epsilon$ for every $f \in \mathcal{B}$, i.e. that $\mathcal{B}$ is equicontinuous. □

    Remark. There is no reason that we need $A \subset \mathbb{R}^m$. A can be any compact metric space.