Assignment 6: Some Textbook Problem Solutions

18. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

We define $f_n(x) = \frac{1}{n} \times f(x)$. Each f_n is discontinuous and since $0 \leq f(x) \leq 1$, we have that $0 \leq f_n(x) \leq \frac{1}{n}$ for all $x \in \mathbb{R}$. Hence, f_n is a sequence of discontinuous functions approaching the continuous function $g(x) \equiv 0$ uniformly.

20 c.

Lemma. *If f is differentiable at x and $a_n < x < b_n$ are sequences such that $a_n, b_n \to x$, then* $\frac{f(b_n)-f(a_n)}{b_n-a_n} \to f'(x)$.

Proof.

$$\begin{aligned}
\left|\frac{f(b_n)-f(a_n)}{b_n-a_n} - f'(x)\right| &= \left|\frac{b_n-x}{a_n-b_n}\frac{f(b_n)-f(x)}{b_n-x} + \frac{x-a_n}{b_n-a_n}\frac{f(x)-f(a_n)}{x-a_n} - f'(x)\right| \\
&= \left|\frac{b_n-x}{a_n-b_n}\left(\frac{f(b_n)-f(x)}{b_n-x} - f'(x)\right) + \frac{x-a_n}{b_n-a_n}\left(\frac{f(x)-f(a_n)}{x-a_n} - f'(x)\right)\right| \\
&\leq \max\left\{\left|\frac{f(b_n)-f(x)}{b_n-x} - f'(x)\right|, \left|\frac{f(x)-f(a_n)}{x-a_n} - f'(x)\right|\right\} \\
&\to 0.
\end{aligned}$$

□

Now, suppose $a_N = \frac{k}{4^N} \leq x < \frac{k+1}{4^N} = b_N$.

$$\frac{f(a_N)-f(b_N)}{a_N-b_N} = \sum_{n=0}^{\infty} \frac{g(4^n a_N) - g(4^n b_N)}{4^{n-N}}.$$

- If $n \geq N$, then $4^n a_N - 4^n b_N$ is an integer. Since g is 1-periodic, this implies that $g(4^n a_N) - g(4^n b_N) = 0$.
- If $n < N$, then there cannot be a half integer $\frac{j}{2}$ lying strictly between $4^n a_N$ and $4^n b_N$ because then $\frac{k}{4^{N-n}} < \frac{2j4^{N-n-1}}{4^{N-n}} < \frac{k+1}{4^{N-n}}$, i.e. $k <$ integer $< k+1$, which is impossible. Hence, $4^n a_N$ and $4^n b_N$ lie on the same "sawtooth line segment" whence $g(4^n a_N) - g(4^n b_N) = \pm 4^n b_N - 4^n a_N = \pm 4^{n-N}$.

Thus, it follows that

$$\frac{f(a_N)-f(b_N)}{a_N-b_N} = \sum_{n=0}^{N-1} \pm 1 \equiv N(mod\ 2).$$

Since this is an integer which changes parity with N, the limit of this sequence cannot possibly exist. Thus, by our above lemma, it follows that $f'(x)$ does not exist. Since x was arbitrary, it follows that f is a continuous, nowhere differentiable function. □

24. By definition

$$cl(C)^c := \left(\bigcap_{\substack{A \supset C \\ A \text{ closed}}} A \right)^c = \bigcup_{\substack{A \supset C \\ A \text{ closed}}} A^c = \bigcup_{\substack{U \subset C^c \\ U \text{ open}}} U.$$

Hence, $cl(C) = M \iff cl(C)^c = \emptyset \iff U \not\subset C^c$ for any non-empty open set $U \iff$ each non-empty open set U contains some $c \in C \iff$ each U_i contains some $c \in C$ (because each U contains some U_i). Hence, if we simply choose one element $c_i \in U_i$ for each i, then $C = \{c_i\}_{i=1}^{\infty}$ is a countable set with $cl(C) = M$. $\square$