Assignment 5: Some Textbook Problem Solutions

34. Let $T(\theta) : \frac{\mathbb{R}}{2\pi\mathbb{Z}} \to \mathbb{R}$ denote the temperature at longitude (angle parameter) $\theta (mod\ 2\pi)$ on the equator. By assumption, this is a continuous function. Thus, $g(\theta) = T(\theta) - T(\theta + \pi)$ is a continuous function with $g(0) = T(0) - T(\pi) = -(T(\pi) - T(2\pi)) = -g(\pi)$. Thus, $g(0)$ and $g(\pi)$ are real numbers with opposite signs. By the intermediate value theorem, there is some $0 \leq \alpha \leq \pi$ with $g(\alpha) = 0$, i.e. α and $\alpha + \pi$ are diametrically opposite points on the equator with the same temperature. $\square$

    Remark. The above argument shows that we can find antipodal points on ANY great circle with the same temperature. One might wonder whether this extra parameter (of varying the great circle) gives us the ability to prove something more spectacular. In fact, we can: there are two antipodal points on the Earth with the same barometric pressure and the same temperature. This is a consequence of the Borsuk-Ulam Theorem.

37. Replacing f with $-f$ if necessary, we may assume that $f'(a) > 0, f'(b) < 0$. By definition, we can find some $\delta > 0$ such that if $0 < h < \delta$, then $\left| \frac{f(a+h)-f(a)}{h} - f'(a) \right| < |f'(a)|$. Hence, for all $0 < h < \delta, f(a+h) > f(a)$. Hence, a is not a local maximum. A completely analogous argument shows that b is not a local maximum. Thus, since f is differentiable, and in particular continuous, it must attain a local (or even a global) maximum for some $x_0 \in (a, b)$. But for such an x_0, we know that $f'(x_0) = 0$. $\square$

39. It is implicit in the statement that f is twice differentiable on (a, b). Consider $g(x) = x^2 f(x)$. Differentiating twice gives that $g''(x) = x^2 f''(x) + 4xf'(x) + 2f(x) \geq 0$ by assumption. Also, since $f(a) = f(b) = 0$, we know that $g(a) = g(b) = 0$.
    Suppose $g(c) > 0$ for some $c \in [a, b]$. Then g attains a local maximum for some $x_0 \in (a, b)$, so in particular, $g(x_0) \geq g(c) > 0$. At this point, $g'(x_0) = 0$. But $g''(x) \geq 0$ for all $x \in (a, b)$, implying that g' is a non-decreasing function. Hence, $g'(x) \geq 0$ for all $x \geq x_0$, which implies that $g(x) \geq g(x_0)$ for all $x \geq x_0$, implying that $0 = g(b) \geq g(x_0) > 0$, a contradiction. Hence, $g(c) \leq 0$ for all $c \in [a, b]$, implying that $f(c) \leq 0$ for all $c \in [a, b]$.

    Remark. The second paragraph of the above proof actually shows the seemingly stronger result of problem 38. in the text: If f is continuous on $[a, b]$ and is twice differentiable on (a, b) with $f''(x) \geq 0$ for all $x \in (a, b)$ then f is convex, i.e. it satisfies the inequality

    $$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

    for all $x, y \in [a, b], 0 \leq t \leq 1$. This is nice because convexity is a very useful property (e.g. Jensen's Inequality) and $f''(x) \geq 0$ is often simple to verify in practice.