Assignment 4: Some Textbook Problem Solutions

5. a. $\bigcup_{n=2}^{\infty} B_{1-1/n}(0) = B_1(0)$ is an open cover of $B_1(0)$ with no finite subcover.

   b. $\bigcup_{n=1}^{\infty}(-n,n) \supset \mathbb{Z}$ is an open cover of $\mathbb{Z}$ with no finite subcover.

17. – Pick any $x_0 \in K$. Let $R = d(x_0, x)$. Certainly, any candidate for the minimum distance to x must lie in $K' = B_R[x] \cap K$, (where square brackets denote a closed ball). More precisely, $\inf_{z \in K} d(x,z) = \inf_{z \in K'} d(x,z)$. But the set K' is closed and bounded in $\mathbb{R}^n$ and so is compact. Also, the function $f : \mathbb{R}^n \to \mathbb{R} : z \mapsto d(x,z)$ is a continuous, since $|d(x,z) - d(x,z')| \leq d(z,z')$ by the triangle inequality. Hence, it achieves its minimum on the compact set K'. That is, there is some $y \in K' \subset K$ for which

$$d(x,y) = f(y) = \inf_{z \in K'} d(x,z) = \inf_{z \in K} d(x,z).$$

    – $K = (0,\infty) \subset \mathbb{R}$ is open and $0 \in \mathbb{R} - K$. But clearly, the minimum distance of 0 is not achieved.

    – The result is not even true in general metric spaces for K closed, as might be suspected from our crucial use of the fact that closed and bounded implies compact. For example, l^∞ is a (complete) metric space and for

$$x_n(i) = \begin{cases} 0 & \text{if } i \neq n, \\ 1 + 1/n & \text{if } i = n. \end{cases}$$

    we see that the set $K = \{x_1, ..., x_n, ...\}$ is closed, $0 \notin K$, and the minimum distance of 1 is not achieved for any point of K. □

20. Let $x \in X - A$. For any $y \in A$, there are disjoint open sets $U_y, V_y \subset X$ such that $x \in V_y$ and $y \in U_y$ (for example, $U_y = B_r(y), V_y = B_r(x)$ where $0 < r < d(x,y)/2$.). $\{U_y\}_{y \in A}$ is certainly an open cover of A. Hence, we can find a finite subcover, say $U = \bigcup_{i=1}^{n} U_{y_i} \supset A$. Then U is an open set which is disjoint from the open set $x \in V_x = \bigcap_{i=1}^{n} V_{y_i}$, by construction. In particular, the open set V is disjoint from A. It follows that $X - A = \bigcup_{x \in X - A} V_x$ is open. □

    Remark. The same proof shows that any compact subset of a Hausdorff space is closed. However, compact does not imply close for an arbitrary topological space, e.g. the indiscrete topology on any set with 2 or more points.

32. We show more generally that if $||x_{n+1} - x_n|| \leq a_n$ for some sequence of positive numbers a_n with $\sum_n a_n < \infty$, then x_n converges.
    Since the sum is convergent, we can find N such that $\sum_{n \geq N} a_n < \epsilon$. For any $m \geq m \geq N$,

$$||x_m - x_n|| \geq \sum_{i=n}^{m-1} ||x_{i+1} - x_i|| \leq \sum_{i \geq N} a_i < \epsilon.$$

    Hence, x_n is Cauchy and so converges.
    Since $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1 < \infty$, the given problem is a special case. □

33. If $\mathbb{R}^n = \bigcup_{k=1}^{\infty} A_k$ iff $\mathbb{R}^n = \bigcup_{k=1}^{\infty} B_k$, and A_k is nowhere dense iff $\operatorname{int}(B_k) = \emptyset$. Also, $\mathbb{R}^n = \bigcup_{k=1}^{\infty} B_k$ iff $\emptyset = \bigcap_{k=1}^{\infty} U_k$, where $U_k = \mathbb{R}^n - B_k$ is a dense open set. Thus, we prove that $\bigcap_{k=1}^{\infty} U_k \neq \emptyset$.

    – Since U_1 is dense, it is certainly non-empty. Thus, we can find some $B_{r_1}[x_1] \subset U_1$.

- Suppose we have constructed $B_{r_N}[x_N] \subset B_{r_{N-1}}[x_{N-1}] \subset ... \subset B_{r_1}[x_1]$. with $r_k \leq 1/k$. Since U_{N+1} is dense and $B_{r_N}(x_N)$ is an open set, we can find some $x_{N+1} \in U_{N+1} \cap B_{r_N}(x_N)$. Since U_{N+1} is open, we can find some $0 < s_{N+1} < 1/N+1$ such that $B_{s_{N+1}}(x_{N+1}) \subset U$. Letting $r_{N+1} = s_{N+1}/2 < 1/N+1$, we get that $B_{r_{N+1}}[x_N] \subset B_{r_N}[x_{N-1}] \subset ... \subset B_{r_1}[x_1]$ with $B_{r_k}[x_k] \subset U_k, r_k < 1/k$ for each $1 \leq k \leq N+1$.

Thus, we have inductively defined a sequence $B_{r_k}[x_k]$ of nested closed balls with $r_k \to 0$. Choose N such that $r_N < \epsilon/2$. For all $m', m \geq M, x_{m'}, x_m \in B_{r_N}[x_N]$. Hence,

$$d(x_{m'}, x_m) \leq d(x_{m'}, x_N) + d(x_m, x_N) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence, $\{x_n\}$ is a Cauchy sequence in the complete metric space $\mathbb{R}^n$. Thus, it converges to some $x \in \mathbb{R}^n$. In fact,

$$x \in \text{cl}(\{x_m, x_{m+1}, ...\}) \subset B_{r_m}[x_m] \subset U_m$$

for all m. Hence, $x \in \bigcap_{k=1}^{\infty} U_k$, proving the theorem. □

Remark. The EXACT same proof carries over to show that any complete metric space (X, d) satisfies the Baire Category Theorem.

39. See Assignment 2 solutions for proofs of these properties of the Cantor set. The Cantor set is a very worthwhile object to think about, so please do so.