Assignment 3: Some Textbook Problem Solutions

23. Let $I = \bigcup_{U \text{open} \subset A} U$.

    - If $x \in \text{int}(A)$, then there is some open ball $B_r(x) \subset A$. $B_r(x)$ is an open set and so is contained in the above union, i.e. $x \in I$.
    - If $x \in I$, then $x \in U$ for some open set $U \subset A$. Then since U is open, there is some $r > 0$ such that $B_r(x) \subset U$. Hence, $x \in \text{int}(A)$, by definition.

    It follows that $I = \text{int}(A)$, as required. □

26. It is easily seen that $a_n \geq 1$ for all n. Observe that if a limit L of this sequence exists, we must have $L = 1 + \frac{1}{1+L}$, implying that $L = \sqrt{2}$. Note that

$$\begin{aligned} a_n - \sqrt{2} &= \frac{2 + a_{n-1}}{1 + a_{n-1}} - \sqrt{2} \\ &= \frac{(2 - \sqrt{2}a_{n-1}) + (a_{n-1} - \sqrt{2})}{1 + a_{n-1}} \\ &= \frac{1 - \sqrt{2}}{1 + a_{n-1}}(a_{n-1} - \sqrt{2}). \end{aligned}$$

Hence, $|a_n - \sqrt{2}| \leq \frac{\sqrt{2}-1}{2}|a_{n-1} - \sqrt{2}|$, where $r = \frac{\sqrt{2}-1}{2} < 1$. Iterating this inequality gives that $|a_n - \sqrt{2}| \leq r^n|\sqrt{2} - a_0| = r^n|\sqrt{2} - 1|$. Hence, $a_n \to \sqrt{2}$. □

34. The given inequality allows us to prove that the sequence $\{x_n\}$ is Cauchy.
    Iterating the given inequality shows that $d(x_{n+1}, x_n) \leq Cr^n$ where $C = d(x_1, x_0)$.
    Suppose $m \geq n > N$.

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{m-1} Cr^{i-1} \\ &\leq \sum_{i=N+1}^{\infty} Cr^{i-1} \\ &= Cr^N/(1-r). \end{aligned}$$

Since $0 \leq r < 1, r^N \to 0$. Thus, we can choose $N = N(\epsilon)$ sufficiently large so that $0 \leq Cr^N/(1-r) < \epsilon$. Then $d(x_m, x_n) < \epsilon$ for all $N \geq N(\epsilon)$. Hence, $\{x_n\}$ is a Cauchy sequence in the complete metric space $\mathbb{R}^n$ and so the sequence converges. □

43. Each $x_n > 0$. Hence,

$$
\begin{aligned}
|x_{n+1} - x_n| &= |\sqrt{3 + x_n} - \sqrt{3 + x_{n-1}}| \\
&= |\frac{(3 + x_n) - (3 + x_n)}{\sqrt{3 + x_n} + \sqrt{3 + x_{n-1}}}| \\
&= \frac{1}{2\sqrt{3}}|x_n - x_{n-1}|.
\end{aligned}
$$

Since $0 \leq \frac{1}{2\sqrt{3}} < 1$, problem 34 gives that x_n converges, say $x_n \to L$. Then by continuity of the function $f(x) = \sqrt{3 + x}$ on $(0, \infty)$, we see that

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{3 + x_{n-1}} = \sqrt{3 + L}.$$

Thus, $L^2 = 3 + L$. This has a unique non-negative solution $L = \frac{1+\sqrt{13}}{2}$ and so

$$x_n \to \frac{1 + \sqrt{13}}{2}.$$