Assignment 2: Some Textbook Problem Solutions

30. Let $f(x) = x$ be the identity function and $g(x) = 1$ a constant function.

    – $||f - g||_\infty = 1$.

    – $||f + g||_\infty = 2$.

    – $||f||_\infty = 1$.

    – $||g||_\infty = 1$.

    Hence

    $$||f - g||_\infty^2 + ||f + g||_\infty^2 = 1 + 4 \neq 2 + 2 = 2||f||_\infty^2 + 2||g||_\infty^2,$$

    proving that the parallelogram law fails. Since the parallelogram law holds for any norm induced by an inner product, it follows that $||.||_\infty$ is not induced by an inner product. □

For the next two problems, we will use the following lemma:

Lemma..

$$\lim_{x\to\infty} \frac{\log(x)}{x} = 0.$$

Proof. Note that

- $\frac{\log(x)}{x}$ is bounded for $x > 1$. Indeed, $g(x) = x - \log(x)$ has a positive derivative for all $x \geq 1$ and has $g(1) > 0$. Hence, g is positive for all $x \geq 1$, implying that $\frac{\log(x)}{x} \leq 1$ for all $x > 1$.
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$$0 \leq \frac{\log(x)}{x} = \frac{2\log(\sqrt{x})}{\sqrt{x}\sqrt{x}} = \left(\frac{\log(\sqrt{x})}{\sqrt{x}}\right) \times \frac{2}{\sqrt{x}} \leq \frac{2}{\sqrt{x}}.$$

By the Sandwich Lemma, it follows that $\lim_{n\to\infty} \frac{\log(x)}{x} = 0$, as desired. □

Note also that this implies that $\lim_{y\to\infty} \frac{y}{e^{cy}} = 0$ and $\lim_{n\to\infty} \frac{e^{cy}}{y} = \infty$ for any positive constant $c > 0$.

33b. $n^{1/n} = e^{\log(n)/n}$. By the above limit evaluation, $\lim_{n\to\infty} \log(n)/n = 0$. Thus, since $f(y) = e^y$ is a continuous function, it follows that

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} f(\log(n)/n) = f(\lim_{n\to\infty} \log(n)/n) = f(0) = 1.$$

45. $|z^n/n| = |z|^n/n = e^{cn}/n$, where $c\ \ \log(|z|) > 0$ since $|z| > 1$. Hence, by the limit lemma above,

$$\lim_{n\to\infty} z^n/n = \infty.$$